December 15, 1892.

Sir JOHN EVANS, K.C.B., D.C.L., LL.D., Treasurer, in the Chair.

A List of the Presents received was laid on the table, and thanks ordered for them.

The Right Hon. John Morley, a member of Her Majesty's Most Honourable Privy Council, whose certificate had been suspended as required by the Statutes, was balloted for and elected a Fellow of the Society.

The following Papers were read:—

I. "On an Apparatus for facilitating the Reduction of Tidal Observations." By G. H. DARWIN, F.R.S., Plumian Professor and Fellow of Trinity College, Cambridge. Received November 12, 1892.

§ 1. *Introduction.*

The tidal oscillation of the ocean may be represented as the sum of a number of simple harmonic waves which go through their periods approximately once, twice, thrice, four times in a mean solar day. But these simple harmonic waves may be regarded as being rigorously diurnal, semi-diurnal, ter-diurnal, and so forth, if the length of the day referred to be adapted to suit the particular wave under consideration. The idea of a series of special scales of time is thus introduced, each time-scale being appropriate to a special tide. For example, the mean interval between successive culminations of the moon is 24^h 50^m, and this interval may be described as the mean lunar day. Now there is a series of tides, bearing the initials M_1, M_2, M_3, M_4, &c., which go through their periods rigorously once, twice, thrice, four times, &c., in a mean lunar day. The solar tides, S, proceed according to mean solar time; but, besides mean lunar and mean solar times, there are special time scales appropriate to the larger (N) and smaller (L) lunar elliptic tides, to the evectional (ν), to the diurnal (K_1) and semi-diurnal (K_2) luni-solar tides, to the lunar diurnal (O), &c.

The process of reduction consists of the determination of the mean height of the water at each of 24 special hours, and subsequent harmonic analysis. The means are taken over such periods of time that the influence of all the tides governed by other special times is eliminated.

The process by which the special hourly heights have hitherto been obtained is the entry of the heights observed at the mean solar hours in a schedule so arranged that each entry falls into a column appropriate to the nearest special hour. Schedules of this kind were prepared by Mr. Roberts for the Indian Government.* The successive rearrangements for each sort of special time were made by recopying the whole of the observations time after time into a series of appropriate schedules. The mere clerical labour of this work is enormous, and great care is required to avoid mistakes.

All this copying might be avoided if the observed heights were written on movable pieces. But a year of observation gives 8,760 hourly heights, and the orderly sorting and re-sorting of nearly 9,000 pieces of paper or tablets might prove more laborious and more treacherous than recopying the figures.

It occurred to me, however, that the marshalling of movable pieces might be reduced to manageable limits if all the 24 observations pertaining to a single mean solar day were moved together, for the movable pieces would be at once reduced to 365, and each piece might be of a size convenient to handle.

The realisation of this plan affords the subject of this paper, and it will appear that not only is all desirable accuracy attainable, but that the other requisite of such a scheme is satisfied, namely, that the whole computing apparatus shall serve any number of times and for any number of places.

The first idea which naturally occurred was to have narrow sliding tablets which should be thrown into their places by a number of templates. It is unnecessary to recount all my trials and failures, but it will suffice to say that the slides and templates require the precision of a mathematical instrument if they are to work satisfactorily, and that the manufacture would be so expensive as to make the price of the instrument prohibitive.

The idea of making the tablets or strips to slide into their places was then abandoned, and the strips are now made with short pins on their under sides, so that they can be stuck on to a drawing board in any desired position. The templates, which were also troublesome to make, are replaced by large sheets of paper with numbered marks on

* An edition of these computation forms was reprinted by aid of a grant from the Royal Society, and is sold by the Cambridge Scientific Instrument Company, but only about a dozen copies now remain. In the course of the preparation of the "guide sheets" of the method proposed in this paper, I found that there are many small mistakes in these Indian forms, but they are fortunately not of such a kind as to produce a sensible vitiation of results. I learn that the mistakes arose from a misunderstanding on the part of a computer employed to draw up the forms.

The accuracy of my guide sheets was controlled by aid of Mr. Roberts's forms, and it was the occasional discrepancy between my results and the forms which led to the detection of the errors referred to.

them to show how the strips are to be set. The guide sheet is laid on a drawing board, and the pins on the strips pierce the paper and fix them in their proper positions.

The shifting of the strips from one arrangement to the next is certainly slower than when they slid into their places automatically, but I find that even without practice it only takes about 7 or 8 minutes to shift 74 of them from any one arrangement to a new one.

The strip belonging to each mean solar day is divided by black lines into 24 equal spaces, intended for the entry of the hourly heights of water. The strip is 9 in. long by $\frac{1}{5}$ in. wide and the divisions ($\frac{3}{8}$ by $\frac{1}{5}$) are of convenient size for the entries. There was much difficulty in discovering a good material, but after various trials artificial ivory, or xylonite, was found to serve the purpose. Xylonite is white, will take writing with Indian ink or pencil, and can easily be cleaned with a damp cloth. It is just as easy to write with liquid Indian ink as with ordinary ink, which must not be used, because it stains the surface.

The strips have a great tendency to warp, and I have two methods of overcoming this. A veneer of xylonite on hard wood serves well, or solid xylonite may be stiffened by sheet brass let into a slot on the under side. In the first plan the pins are fixed in the wood, and in the second the brass is filed to a spike at each end. Whichever plan is adopted, the strips are expensive, costing about £7 for a set, and I do not at present see any way of making them cheaper.

The observations are to be treated in groups of two and a half lunations or 74 days. A set of strips, therefore, consists of 74, numbered from 0 to 73 in small figures on their flat ends.

If a set be pinned horizontally on a drawing board in vertical column, we have a form consisting of rows for each mean solar day and columns for each hour. The observed heights of the water are then written on the strips.

When the 24 columns are summed and divided by the number of entries we obtain the mean solar hourly mean heights. The harmonic analysis of these means gives the mean solar tides. But for evaluating the other tides the strips must be rearranged, and to this point we turn our attention.

Let us consider a special case, that of mean lunar time. A mean lunar hour is about $1^h\ 2^m$ m.s. time; hence the 12^h of each m.s. day must lie within 31^m m.s. time of a mean lunar hour. The following sample gives the incidence to the nearest lunar hour of the first few days in a year:—

Mean solar time.		=	Mean lunar time.	
0^d	12^h	=	0^d	12^h
1	12	=	1	11
2	12	=	2	10
3	12	=	3	9
4	12	=	4	8
5	12	=	5	8
6	12	=	6	7
7	12	=	7	6
&c.			&c.	

The successive 12^h of m.s. time will march retrogressively through all the 24 hours of m. lunar time.

Now, if starting from strip 0, we push strip 1 one division to the left, strip 2 two divisions to the left, and so on, the entries on the strips will be arranged in columns of approximately lunar time.

The rule for this arrangement is given by marks on a sheet of paper 18 in. broad; these marks consist of parallel numbered steps or zigzags showing where the ends of each strip are to be placed so as to bring the hourly values into their proper places.

At the end of a lunation mean solar time has gained a whole day over mean lunar time and the 12^h solar again agrees with the 12^h lunar. On the guide sheet we see that the zigzag which takes its origin at the left end of strip 0 has descended diagonally from right to left until it has reached the left margin of the paper, and a new zigzag then begins on the right margin.

When the strips are pinned out following the zigzags on the sheet marked M, the entries are arranged in 48 columns, but the number of entries in each column is different. The 48 columns are to be regarded as appertaining to $0^h, 1^h, \ldots, 22^h, 23^h, 0^h, 1^h, \ldots, 22^h, 23^h$. Thus, the number of entries in the left-hand column of any hour added to the number of entries in the right-hand column of the same hour is, in each case, 74. The 48 incomplete columns may, in fact, be regarded as 24 complete ones.

The 24 complete columns are then summed; the 24 sums would, if divided by the total number of strips, give the 24 mean lunar hourly heights. The harmonic analysis of these sums gives certain constants, which, when divided by the number of strips, are the required tidal constants. It must be remarked, however, that, as the incidence of the entries is not exact in lunar time, investigation must be made of the corrections arising out of this inexactness.

The explanation of the guide sheet for lunar time will serve, *mutatis mutandis*, for all the others.

The zigzags have to be placed so as to bring the columns into exact alignment, and printers' types provide all the accuracy requisite.

Accordingly, the computing strips are made to suit a chosen type. The standard length for one of the 24 divisions on the strips was chosen as that of a "2-em English quadrate"; 24 of these come to 9 inches, which is the length of a strip. I found the English quadrate a little too narrow, and accordingly between each line of quadrates there is a "blind rule," of 42 to the inch. The depth of the guide sheet is that of 74 quadrates and 74 rules, making $15\frac{1}{2}$ in. The computing strips are $\frac{1}{5}$ in. broad, and 74 of them occupy $14\frac{4}{5}$ in. The excess of $15\frac{1}{2}$ above $14\frac{4}{5}$, or $\frac{7}{10}$ in., is necessary to permit the easy arrangement of the strips.

To guard against the risk of the computer accidentally using the wrong sheet, the sheets are printed on coloured paper, the sequence of colours being that of the rainbow. The sheets for days 0 to 73 are all red; those for days 74 to 74+73, or 147, are all yellow; those for days 148 to 148+73, or 221, are green; those for days 222 to 222+73, or 295, are blue; and those for days 296 to 296+73, or 369, are violet.

Thus, when the observations for the first 74 days of the year are written on the strips all the sheets will be red; the strips will then be cleaned, and the observations for the second 74 days written in, when all the guide sheets will be yellow, and so on.

I must now refer to another considerable abridgment of the process of harmonic analysis. It is independent of the method of arrangement just sketched.

In the Indian computation forms the mean solar hourly heights have been found for the whole year, and the observations have been rearranged for the evaluation of certain other tides governed by a time scale which differs but little from the mean solar scale. I now propose to break the mean solar heights into sets of 30 days, and to analyse them, and next to harmonically analyse the 12 sets of harmonic constituents for annual and semi-annual inequalities. By this plan the harmonic constants for 11 different tides are obtained by one set of additions. In fact, we now get the annual, semi-annual, and solar elliptic tides, which formerly demanded much troublesome extra computation. A great saving is secured by this alone, and the results are in close agreement with those derived from the old method.

The guide sheets marked S and the computation forms are arranged so that the observations are broken up into the proper groups of 30 days, and they show the computer how to make the subsequent calculations.

I have also devised an abridged method of evaluating the tides of long period MSf, Mf, Mm. The method is less accurate than that followed hitherto, but it appears to give fairly good results, and reduces the work to very small dimensions.

Before entering on the details of my plan it is proper to mention that Dr. Börgen has devised and used a method for attaining the same end. He has prepared sheets of tracing paper with diagonal lines on them, so arranged that when any sheet is laid on the copy of the observations written in daily rows and hourly columns, the numbers to be summed are found written between a pair of lines. This plan is excellent, but I fear that the difficulty of adding correctly in diagonal lines is considerable, and the comparative faintness of figures seen through tracing paper may be fatiguing to the eyes. Dr. Börgen's plan is simple and inexpensive, and had I not thought that the plan now proposed has considerable advantages I should not have brought it forward.

In the investigations which follow the notation of the Report of 1883 to the British Association on harmonic analysis is used without further explanation.

§ 2. *Evaluation of* A_0, Sa, Ssa, S_1, S_2, S_4, S_6, T, R, K_2, K_1, P.

The 24 mean solar hourly heights of water are entered in a schedule of 24 columns, with one row for each day, extending to n days; the 24 columns are summed, and the sums divided by n; the 24 means are harmonically analysed; it is required to find from the results the values of the harmonic constituents.

The speed of any one of the tides differs from a multiple of 15° per hour by a small angle; thus, any one of the tides is expressible in the form $H\cos[(15°q-\beta)t-\zeta]$, where q is 0, 1, 2, 3, &c., and β is small.

When t lies between 0^h and 24^h this formula expresses the oscillation of level due to this tide on the day 0 of the series of days.

If multiples of 24^h from 1 to $n-1$ be added to t, the expression gives the height at the same hour, t, of mean solar time on each of the succession of days.

Then if $\mathfrak{h}$ denotes the mean height of water, as due to this tide alone, at the hour t, we have

$$\mathfrak{h} = \frac{1}{n}\sum_1^n H\cos[(15°q-\beta)\,t-\zeta+(15°q-\beta)\,24\,(n-1)]$$

$$= \frac{1}{n}H\frac{\sin 12n\beta}{\sin\beta}\cos[(15°q-\beta)t-\zeta-12\beta\,(n-1)] \quad\ldots\ldots \quad (1).$$

When t is put successively equal to 0^h, 1^h, ..., 23^h we get the 24 values of $\mathfrak{h}$ which are to be submitted to harmonic analysis.

The mean value of $\mathfrak{h}$, say A_0 (not to be confused with A_0 as written at the head of this section, where is denoted the mean sea level above datum) is found by taking the mean of the 24 values of $\mathfrak{h}$.

By the formula for the summation of a series of cosines it is easy to prove that

$$A_0 = \frac{1}{24n} H \frac{\sin 12\,n\beta}{\sin(\frac{1}{2}\beta - \frac{15}{2} q)} \cos\left[\zeta + (12n - \tfrac{1}{2}\beta) + \tfrac{15}{2} q\right] \quad \ldots \quad (2).$$

We will now find the p^{th} harmonic constituents A_p, B_p. By the ordinary rules

$$\left.\begin{matrix} A_p \\ B_p \end{matrix}\right\} = \tfrac{1}{12} \sum_0^{23} \mathfrak{h} \begin{matrix}\cos\\ \sin\end{matrix} 15^\circ pt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Now

$$2 \begin{matrix}\cos\\ \sin\end{matrix} 15^\circ pt \cos\left[(15q - \beta)\,t - \zeta - 12\,\beta\,(n-1)\right]$$

$$= \begin{matrix}\cos\\ \sin\end{matrix} \left[(15^\circ (q+p) - \beta)\,t - \zeta - 12\,\beta\,(n-1)\right]$$

$$+ \begin{matrix}+\cos\\ -\sin\end{matrix} \left[(15^\circ (q-p) - \beta)\,t - \zeta - 12\,\beta\,(n-1)\right];$$

and $\frac{1}{12}$ of the sum of the 24 values corresponding to $t = 0, 1, \ldots, 23$ is

$$\tfrac{1}{24} \frac{\sin 12\left[15^\circ (q+p) - \beta\right]}{\sin \frac{1}{2}\left[15^\circ (q+p) - \beta\right]} \begin{matrix}\cos\\ \sin\end{matrix} \left[\tfrac{23}{2}(15^\circ (q+p) - \beta) - \zeta - 12\,\beta\,(n-1)\right]$$

$\pm$ the same with sign of p changed

This expression admits of simplification, because $12 \times 15^\circ = 180^\circ$; making this simplification, and introducing the result into (3), we obtain

$$\left.\begin{matrix} A_p \\ B_p \end{matrix}\right\} = \frac{1}{24n} H \sin 12\,n\beta \left\{ \frac{\begin{matrix}-\cos\\ +\sin\end{matrix}\left[\zeta + (12\,n - \frac{1}{2})\,\beta + \frac{15}{2}^\circ (q+p)\right]}{\sin\left[\frac{15}{2}^\circ (q+p) - \frac{1}{2}\beta\right]} \right.$$

$$\left. \frac{\begin{matrix}-\cos\\ -\sin\end{matrix}\left[\zeta + (12n - \frac{1}{2})\,\beta + \frac{15}{2}^\circ (q-p)\right]}{\sin\left[\frac{15}{2}^\circ (q-p) - \frac{1}{2}\beta\right]} \right\} \quad \ldots\ldots \quad (4).$$

In the particular case where $p = q$, we have

$$\left.\begin{matrix} A_q \\ B_q \end{matrix}\right\} = \frac{1}{24n} H \sin 12\,n\beta \left\{ \frac{\begin{matrix}-\cos\\ +\sin\end{matrix}\left[\zeta + (12\,n - \frac{1}{2})\,\beta + 15^\circ q\right]}{\sin(15^\circ q - \frac{1}{2}\beta)} \right.$$

$$\left. \frac{\begin{matrix}+\cos\\ +\sin\end{matrix}\left[\zeta + (12\,n - \frac{1}{2})\,\beta\right]}{\sin \frac{1}{2}\beta} \right\} \quad \ldots\ldots\ldots\ldots \quad (5).$$

If the number of days n be large, A_p, B_p will be small unless the denominator of one of the two terms in (4) be very small. This last case can only occur when $p = q$ and when β is small. Hence, in the analysis of a term of the form under consideration, we may neglect all the harmonics except the q^{th} one. Accordingly (2) and (5) are the only formulæ required.

A case, however, which there will be occasion to use hereafter is when $n = 30$, $q = 2$, when (4) becomes

$$\left.\begin{matrix} A_2 \\ B_2 \end{matrix}\right\} = \tfrac{1}{720} H \sin 360\beta \left\{ \frac{\begin{smallmatrix}\cos \\ \sin\end{smallmatrix} (\zeta + 359\tfrac{1}{2}\beta)}{\sin \tfrac{1}{2}\beta} \quad \frac{\begin{smallmatrix}-\cos \\ +\sin\end{smallmatrix} (\zeta + 30^\circ + 359\tfrac{1}{2}\beta)}{\sin (30^\circ - \tfrac{1}{2}\beta)} \right\} \quad (6).$$

For the present we have to apply (5) in the two cases $q = 1$, $\beta = 0^\circ{\cdot}0410686$ and $q = 2$, $\beta = 0^\circ{\cdot}0821372$; now the ratios of cosec $\tfrac{1}{2}\beta$ to cosec $(15^\circ q - \tfrac{1}{2}\beta)$ in these two cases are 722 to 1 and 697 to 1. In both cases the first term of (5) is negligible compared with the second.

Now write
$$\mathfrak{J} = \frac{24n \sin \tfrac{1}{2}\beta}{\sin 12\, n\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7),$$

and (5) becomes, with sufficient exactness,

$$\left.\begin{matrix} A_q \\ B_q \end{matrix}\right\} = \frac{H}{\mathfrak{J}} \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} [\zeta + (12n - \tfrac{1}{2})\beta] \quad \ldots\ldots\ldots\ldots \quad (8).$$

If this be compared with (2), we see that when $q = 0$ this formula also comprises (2).

In the applications to be made β is very small, so that $\mathfrak{J}$ is approximately a function of the form θ cosec θ. This function increases very rapidly when θ passes 90°, but for considerable values less than 90° it only slightly exceeds unity; for example, when $\theta = 60^\circ$, $\mathfrak{J} = 1{\cdot}2$, but when $\theta = 180^\circ$, $\mathfrak{J}$ = infin.

It follows, therefore, that if the number n of days in the series is such that $12n\beta$ is less than say 60°, the magnitudes of A_q, B_q are but little diminished by division by $\mathfrak{J}$; but if $12n\beta$ is nearly 180°, A_q, B_q become vanishingly small.

If the typical tide here considered be the principal lunar tide M_2, and if the number of days be as nearly as possible an exact multiple of a semi-lunation, $12n\beta$ is nearly 180°, and the corresponding A_2, B_2 become very small. No number of whole days can be an exact multiple of a semi-lunation, so that A_2, B_2 corresponding to M_2 cannot be made to vanish completely. For the present they may be treated as negligible, and we return to this point in the next section.

The above investigation shows that in the expression for the whole

oscillation of sea level upon which the proposed analysis is performed all those tides may be omitted from which β is not very small, and also all those whose frequencies are such that the period under consideration $12n\beta$ is nearly 180°.

Since the period under consideration will be a lunation, it follows that, as far as is now material, the general expression for sea level may be written as follows, t denoting mean solar hour angle equal to $15°t$:—

m. w., annual	$A_0 + H_{sa}\cos(h - \kappa_{sa})$
semi-annual	$+H_{ssa}\cos(2h - \kappa_{ssa})$
Solar tides, S_1, S_2........	$+H_{\frac{1}{2}s}\cos(t - \kappa_{\frac{1}{2}s}) + H_s\cos(2t - \kappa_s)$
S_3, S_4........	$+H_{2s}\cos(4t - \kappa_{2s}) + H_{3s}\cos(6t - \kappa_{3s})$
Solar elliptic, T	$+H_t\cos(2t - h + p_{\prime} - \kappa_t)$
R	$+H_r\cos(2t + h - p_{\prime} + \pi - \kappa_r)$
Luni-solar, K_2	$+f''H''\cos(2t + 2h - 2\nu'' - \kappa'')$
K_1	$+f'H'\cos(t + h - \nu' - \frac{1}{2}\pi - \kappa')$
Solar diurnal, P	$+H_p\cos(t - h + \frac{1}{2}\pi - \kappa_p)$ .... (9).

This includes all the tides whose initials are written at the head of this section.

It is now necessary to break up the year into 12 equidistant lunations of 30 days. This can be done by the omission of 5 days in ordinary years, and of 6 days in leap years.

If the days of the year are numbered 0 to 364 (365 in leap year), the twelve months are as follows:—

0, 0^d to 29^d; 1, 30^d to 59^d; *omit* 60^d; 2, 61^d to 90^d; 3, 91^d to 120^d; *omit* 121^d; 4, 122^d to 151^d; 5, 152^d to 181^d; *omit* 182^d; 6, 183^d to 212^d; 7, 213^d to 242; 8, 243^d to 272^d; *omit* 273^d; 9, 274^d to 303^d; 10, 304^d to 333^d; *omit* 334^d; 11, 335^d to 364^d; *in leap year omit* 365^d.

The increments of sun's mean longitude from 0^d 0^h of month 0 up to 0^h of the day numbered 0 of each group of days or month are as follows:—

0, 0°; 1, 30°−0°·431; 2, 60°·124; 3, 90°−0°·306; 4, 120°·249; 5, 150°−0°·182; 6, 180°·373; 7, 210°−0°·057; 8, 240°−0°·488; 9, 270°·068; 10, 300°−0°·364; 11, 330°·191.

Thus if h_0 be the sun's mean longitude at 0^d 0^h of month 0, the sun's mean longitude at 0^d 0^h of month τ is $h_0 + 30°\tau$, with sufficient approximation.

Now let V with appropriate suffix denote the initial "equilibrium argument" at $0^d\ 0^h$ of month 0, so that

$$V_{sa} = h_o,\ V_{ssa} = 2h_o,\ V_t = -h_o + p_{,},\ V_r = h_o - p_{,} + \pi,\ V'' = 2h_o - 2\nu'',$$

$$V' = h_o - \nu' - \tfrac{1}{2}\pi,\ V_p = -h_o + \tfrac{1}{2}\pi;$$

then the general expression (9) for the tide in the month τ becomes

$$A_o + H_{sa}\cos(\eta t + V_{sa} + 30^\circ\tau - \kappa_{sa}) + H_{ssa}\cos(2\eta t + V_{ssa} + 60^\circ\tau - \kappa_{ssa})$$
$$+ H_{\frac{1}{2}s}\cos(15^\circ t - \kappa_{\frac{1}{2}s}) + H_s\cos(30^\circ t - \kappa_s) + H_{2s}\cos(60^\circ t - \kappa_{2s})$$
$$+ H_{3s}\cos(90^\circ t - \kappa_{3s})$$
$$+ H_t\cos[(30^\circ - \eta)t + V_t - 30^\circ\tau - \kappa_t]$$
$$+ H_r\cos[(30^\circ + \eta)t + V_r + 30^\circ\tau - \kappa_r]$$
$$+ \mathrm{f}''H''\cos[(30^\circ + 2\eta)t + V'' + 60^\circ\tau - \kappa'']$$
$$+ \mathrm{f}'H'\cos[15^\circ + \eta)t + V' + 30^\circ\tau - \kappa']$$
$$+ H_p\cos[(15^\circ - \eta)t + V_p - 30^\circ\tau - \kappa_p] \quad \ldots\ldots\ldots\ldots \quad (10).$$

Each of these terms falls into the type $\cos[(15^\circ q - \beta)t - \zeta]$, and β is in every case either $\pm\eta, -2\eta$, or 0.

Now, when harmonic analysis of the mean of 30 days is carried out, coefficients $\mathfrak{F}$ are introduced.

Write therefore

$$\mathfrak{F}_1 = \frac{24 \times 30 \sin\frac{1}{2}\eta}{\sin 360\eta}, \qquad \mathfrak{F}_2 = \frac{24 \times 30 \sin\eta}{\sin 720\eta}.$$

With the known value of η,

$$\log\mathfrak{F}_1 = 0\cdot00483, \qquad \log\mathfrak{F}_2 = 0\cdot01945.$$

In applying the method investigated above, it will be observed that a term of any frequency $15^\circ q - \beta$ only contributes to the harmonic constituent of order q.

Then applying our general rule (8) term by term, and observing that $359\frac{1}{2}\eta = 14^\circ\cdot76$, and $719\eta = 29^\circ\cdot53$, the result may be written as follows:—

$$\mathfrak{A}_o^{(\tau)} = A_o + \frac{H_{sa}}{\mathfrak{F}_1}\cos(\kappa_{sa} - V_{sa} - 30^\circ\tau - 14^\circ\cdot76)$$
$$+ \frac{H_{ssa}}{\mathfrak{F}_2}\cos(\kappa_{ssa} - V_{ssa} - 60^\circ\tau - 29^\circ\cdot53);$$

$$\left.\begin{matrix}\mathfrak{A}_1^{(\tau)}\\ \mathfrak{B}_1^{(\tau)}\end{matrix}\right\} = H_{\frac{1}{2}s}\,{\cos \atop \sin}\,\kappa_{\frac{1}{2}s} + \frac{\mathrm{f}'H'}{\mathfrak{F}_1}\,{\cos \atop \sin}\,(\kappa' - V' - 30^\circ\tau - 14^\circ\cdot76)$$
$$+ \frac{H_p}{\mathfrak{F}_1}\,{\cos \atop \sin}\,(\kappa_p - V_p + 30^\circ\tau + 14^\circ\cdot76);$$

$$\left.\begin{matrix}\mathfrak{A}_2{}^{(\tau)}\\ \mathfrak{B}_2{}^{(\tau)}\end{matrix}\right\} = H_s \begin{matrix}\cos\\ \sin\end{matrix}\,\kappa_s + \frac{H_t}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}\,(\kappa_t - \mathrm{V}_t + 30^\circ\tau + 14^\circ{\cdot}76)$$

$$+\frac{H_r}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}\,(\kappa_r - \mathrm{V}_r - 30^\circ\tau - 14^\circ{\cdot}76)$$

$$+\frac{\mathrm{f}''H''}{\mathfrak{F}_2}\begin{matrix}\cos\\ \sin\end{matrix}\,(\kappa'' - \mathrm{V}'' - 60^\circ\tau - 29^\circ{\cdot}53);$$

$$\left.\begin{matrix}\mathfrak{A}_4{}^{(\tau)}\\ \mathfrak{B}_4{}^{(\tau)}\end{matrix}\right\} = H_{2s}\begin{matrix}\cos\\ \sin\end{matrix}\,\kappa_{2s};\qquad \left.\begin{matrix}\mathfrak{A}_6{}^{(\tau)}\\ \mathfrak{B}_6{}^{(\tau)}\end{matrix}\right\} = H_{3s}\begin{matrix}\cos\\ \sin\end{matrix}\,\kappa_{3s}. \quad\ldots\ldots\ldots\ldots \quad (11).$$

With the meaning of the term month in the present context, the sun has a mean motion of 30° per month, and each of the first five $\mathfrak{A}$'s and $\mathfrak{B}$'s is a function with a constant part and with annual and semi-annual inequalities.

When τ has successively the 12 values 0, 1, ..., 11, we have 12 equidistant values of the $\mathfrak{A}$'s and $\mathfrak{B}$'s. These may be harmonically analysed for annual and semi-annual inequalities.

Suppose that the several coefficients to be determined by harmonic analysis are defined by the following equations:—

$$\mathfrak{A}_0{}^{(\tau)} = A_0 + A_1\cos 30^\circ\tau + B_1\sin 30^\circ\tau + A_2\cos 60^\circ\tau + B_2\sin 60^\circ\tau;$$

$$\left.\begin{matrix}\mathfrak{A}_1{}^{(\tau)}\\ \mathfrak{B}_1{}^{(\tau)}\\ \mathfrak{A}_2{}^{(\tau)}\\ \mathfrak{B}_2{}^{(\tau)}\end{matrix}\right\} = \left.\begin{matrix}\mathrm{C}_0\\ \mathrm{c}_0\\ \mathrm{E}_0\\ \mathrm{e}_0\end{matrix}\right\} + \left.\begin{matrix}\mathrm{C}_1\\ \mathrm{c}_1\\ \mathrm{E}_1\\ \mathrm{e}_1\end{matrix}\right\}\cos 30^\circ\tau + \left.\begin{matrix}\mathrm{D}_1\\ \mathrm{d}_1\\ \mathrm{F}_1\\ \mathrm{f}_1\end{matrix}\right\}\sin 30^\circ\tau + \left.\begin{matrix}\\ \\ \mathrm{E}_2\\ \mathrm{e}_2\end{matrix}\right\}\cos 60^\circ\tau + \left.\begin{matrix}\\ \\ \mathrm{F}_2\\ \mathrm{f}_2\end{matrix}\right\}\sin 60^\circ\tau$$

$$\tfrac{1}{12}\Sigma\mathfrak{A}_4{}^{(\tau)} = A_4,\quad \tfrac{1}{12}\Sigma\mathfrak{A}_4{}^{(\tau)} = B_4,\quad \tfrac{1}{12}\Sigma\mathfrak{A}_6{}^{(\tau)} = A_6,\quad \tfrac{1}{12}\Sigma\mathfrak{B}_6{}^{(\tau)} = B_6 \ldots \quad (12).$$

Then on comparing (12) with (11) we see that:—

$$A_0 = A_0;$$

$$\left.\begin{matrix}A_1\\ B_1\end{matrix}\right\} = \frac{H_{sa}}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}\,(\kappa_{sa} - \mathrm{V}_{sa} - 14^\circ{\cdot}76),\quad \left.\begin{matrix}A_2\\ B_2\end{matrix}\right\} = \frac{H_{ssa}}{\mathfrak{F}_2}\begin{matrix}\cos\\ \sin\end{matrix}\,(\kappa_{ssa} - \mathrm{V}_{ssa} - 29^\circ{\cdot}53);$$

$$\left.\begin{matrix}\mathrm{C}_0\\ \mathrm{c}_0\end{matrix}\right\} = H_{\frac{1}{2}s}\begin{matrix}\cos\\ \sin\end{matrix}\,\kappa_{\frac{1}{2}s},$$

$$\left.\begin{matrix}\mathrm{C}_1\\ \mathrm{D}_1\end{matrix}\right\} = \begin{matrix}+\\ +\end{matrix}\frac{\mathrm{f}'H'}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}\,(\kappa' - \mathrm{V}' - 14^\circ{\cdot}76)\ \begin{matrix}+\\ -\end{matrix}\frac{H_p}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}\,(\kappa_p - \mathrm{V}_p + 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}\mathrm{c}_1\\ \mathrm{d}_1\end{matrix}\right\} = \begin{matrix}+\\ -\end{matrix}\frac{\mathrm{f}'H'}{\mathfrak{F}_1}\begin{matrix}\sin\\ \cos\end{matrix}\,(\kappa' - \mathrm{V}' - 14^\circ{\cdot}76)\ \begin{matrix}+\\ +\end{matrix}\frac{H_p}{\mathfrak{F}_1}\begin{matrix}\sin\\ \cos\end{matrix}\,(\kappa_p - \mathrm{V}_p + 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}\mathrm{E}_0\\ \mathrm{e}_0\end{matrix}\right\} = H_s\begin{matrix}\cos\\ \sin\end{matrix}\,\kappa_s,$$

$$\left.\begin{matrix}E_1\\F_1\end{matrix}\right\} = \frac{+H_t}{-\mathfrak{F}_1}\begin{matrix}\cos\\\sin\end{matrix}(\kappa_t - V_t + 14^\circ{\cdot}76)\ \frac{+H_r}{+\mathfrak{F}_1}\begin{matrix}\cos\\\sin\end{matrix}(\kappa_r - V_r - 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}e_1\\f_1\end{matrix}\right\} = \frac{+H_t}{+\mathfrak{F}_1}\begin{matrix}\sin\\\cos\end{matrix}(\kappa_t - V_t + 14^\circ{\cdot}76)\ \frac{+H_r}{-\mathfrak{F}_1}\begin{matrix}\sin\\\cos\end{matrix}(\kappa_r - V_r - 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}E_2\\F_2\end{matrix}\right\} = \frac{+f''H''}{+\mathfrak{F}_2}\begin{matrix}\cos\\\sin\end{matrix}(\kappa'' - V'' - 29^\circ{\cdot}53);$$

$$\left.\begin{matrix}e_2\\f_2\end{matrix}\right\} = \frac{+f''H''}{-\mathfrak{F}_2}\begin{matrix}\sin\\\cos\end{matrix}(\kappa'' - V'' - 29^\circ{\cdot}53) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (13).$$

From these equations we get

$$\left.\begin{matrix}\frac{1}{2}(C_1 - d_1)\\\frac{1}{2}(c_1 + D_1)\end{matrix}\right\} = \frac{f'H'}{\mathfrak{F}_1}\begin{matrix}\cos\\\sin\end{matrix}(\kappa' - V' - 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}\frac{1}{2}(C_1 + d_1)\\\frac{1}{2}(c_1 - D_1)\end{matrix}\right\} = \frac{H_p}{\mathfrak{F}_1}\begin{matrix}\cos\\\sin\end{matrix}(\kappa_p - V_p + 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}\frac{1}{2}(E_1 - f_1)\\\frac{1}{2}(e_1 + F_1)\end{matrix}\right\} = \frac{H_r}{\mathfrak{F}_1}\begin{matrix}\cos\\\sin\end{matrix}(\kappa_r - V_r - 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}\frac{1}{2}(E_1 + f_1)\\\frac{1}{2}(e_1 - F_1)\end{matrix}\right\} = \frac{H_t}{\mathfrak{F}_1}\begin{matrix}\cos\\\sin\end{matrix}(\kappa_t - V_t + 14^\circ{\cdot}76);$$

$$\left.\begin{matrix}\frac{1}{2}(E_2 - f_2)\\\frac{1}{2}(e_2 + F_2)\end{matrix}\right\} = \frac{f''H''}{\mathfrak{F}_2}\begin{matrix}\cos\\\sin\end{matrix}(\kappa'' - V'' - 29^\circ{\cdot}53) \ldots\ldots \quad (14).$$

The tidal constants of the several tides enumerated at the head of this section are determinable from these equations.

Our rule is accordingly to analyse in twelve groups of 30 days, and then to analyse the resulting harmonic constituents for annual and semi-annual inequalities, combining the final results according to the formula just found.

The edition of "guide sheets" and computation forms which I have drawn up are so arranged as to facilitate the whole process and to render it quite straightforward. By this single set of additions it is thus possible to evaluate eleven tides and mean water.

§ 3. *Clearance of* T, R *from perturbation by* M_2.

The method of the last section was designed to render all the tides insensible excepting those enumerated. But M_2 is so much larger than any other tide that there is a small residual disturbance which ought to be corrected.

I have made computations, which I do not give, but which show that the disturbance of all the harmonic constituents except $\mathfrak{A}_2$, $\mathfrak{B}_2$

is insensible. It is required then to determine the correction to be applied to $\mathfrak{A}_2$, $\mathfrak{B}_2$, and thence those for E, F, e, f.

Suppose that, when time t is counted from $0^d\ 0^h$ of month τ, the M_2 tide is expressed by $M\cos[(30°-\beta)t-\zeta]$, where $\beta = 1°{\cdot}0158958$.

When means taken over 30 days are harmonically analysed the formula (6) gives the contributions to $\mathfrak{A}_2$, $\mathfrak{B}_2$. As it is now required to obliterate these contributions, the signs must be changed, and the corrections are

$$\left.\begin{matrix}\delta\mathfrak{A}_2^{(\tau)}\\ \delta\mathfrak{B}_2^{(\tau)}\end{matrix}\right\} = -\tfrac{1}{720}M\sin 360\beta\left\{\frac{\begin{matrix}\cos\\ \sin\end{matrix}(\zeta+359\tfrac{1}{2}\beta)}{\sin\tfrac{1}{2}\beta}\ \frac{\begin{matrix}-\cos\\ +\sin\end{matrix}(\zeta+30°+359\tfrac{1}{2}\beta)}{\sin(30°-\tfrac{1}{2}\beta)}\right\} \quad \ldots\ldots \quad (15).$$

For reasons which will appear below I now write

$$\zeta = \zeta_m^{(\tau)}-0°{\cdot}5258.$$

Then introducing the value of β into (15), I find

$$\left.\begin{matrix}\delta\mathfrak{A}_2^{(\tau)}\\ \delta\mathfrak{B}_2^{(\tau)}\end{matrix}\right\} = -\tfrac{1}{720}M\sin 5°\ 43'{\cdot}35\left\{\frac{\begin{matrix}\cos\\ \sin\end{matrix}(\zeta_m^{(\tau)}+4°{\cdot}689)}{\sin 0°\ 30'\ 28''{\cdot}6}\ \frac{\begin{matrix}-\cos\\ +\sin\end{matrix}(\zeta_m^{(\tau)}+34°{\cdot}689)}{\sin 29°\ 29'{\cdot}52}\right\} \quad \ldots \quad (16).$$

Let ζ_m denote the value of $\zeta_m^{(\tau)}$ at $0^d\ 0^h$ of month 0, and let $\zeta_m^{(\tau)} = \zeta_m+\theta^{(\tau)}$, and let $\mathfrak{f}_2$ denote a certain factor whose logarithm is 0·00849, and let $M = \mathrm{f}H_m$.

In the harmonic analysis for the M_2 tide, considered below in § 6, we shall have

$$A_2 = \frac{\mathrm{f}H_m}{\mathfrak{f}_2}\cos\zeta_m, \qquad B_2 = \frac{\mathrm{f}H_m}{\mathfrak{f}_2}\sin\zeta_m.$$

Accordingly

$$M\begin{matrix}\cos\\ \sin\end{matrix}\zeta_m^{(\tau)} = \mathfrak{f}_2\left[A_2\begin{matrix}\cos\\ \sin\end{matrix}\theta^{(\tau)}\mp B_2\begin{matrix}\sin\\ \cos\end{matrix}\theta^{(\tau)}\right].$$

These values of $M\begin{matrix}\cos\\ \sin\end{matrix}\zeta_m^{(\tau)}$ must now be introduced into (16), but the algebraic process need not be given in detail. If we write

$$\left.\begin{matrix}\tfrac{1}{2}(\mathrm{S}+\mathrm{P})\\ \tfrac{1}{2}(\mathrm{R}+\mathrm{Q})\end{matrix}\right\} = \tfrac{1}{720}\mathfrak{f}_2\frac{\sin 5°\ 43'{\cdot}35\begin{matrix}\cos\\ \sin\end{matrix}(4°\ 41'{\cdot}32)}{\sin 0°\ 30'\ 28''{\cdot}6},$$

$$\left.\begin{matrix}\tfrac{1}{2}(\mathrm{S}-\mathrm{P})\\ \tfrac{1}{2}(\mathrm{R}-\mathrm{Q})\end{matrix}\right\} = \tfrac{1}{720}\mathfrak{f}_2\frac{\sin 5°\ 43'{\cdot}35\begin{matrix}\cos\\ \sin\end{matrix}(34°\ 41'{\cdot}32)}{\sin 29°\ 29'\ 31''{\cdot}4},$$

it follows that

$$P = 0{\cdot}01564, \quad Q = 0{\cdot}00114, \quad R = 0{\cdot}00147, \quad S = 0{\cdot}01611.$$

Then, when the substitution of the values of $M \frac{\sin}{\cos} \zeta_m^{(\tau)}$ is carried out, we find

$$\left.\begin{matrix}\delta\mathfrak{A}_2^{(\tau)} \\ \delta\mathfrak{B}_2^{(\tau)}\end{matrix}\right\} = \cos\theta^{(\tau)}\left[\begin{matrix}-P \\ -R\end{matrix} A_2 \begin{matrix}+Q \\ -S\end{matrix} B_2\right] + \sin\theta^{(\tau)}\left[\begin{matrix}+Q \\ -S\end{matrix} A_2 \begin{matrix}+P \\ +R\end{matrix} B_2\right].$$

By the definition of $\theta^{(\tau)}$ it appears that $-\theta^{(\tau)}$ is the increment of twice the mean moon's hour angle during the time from $0^d\ 0^h$ of month 0 up to $0^d\ 0^h$ of month τ, that is to say $\theta^{(\tau)} = -2(\gamma - \sigma)t$ for the time specified. The following table gives the values of $\theta^{(\tau)}$ and of its cosine and sine for each month :—

Month (τ).	No. of days from epoch 0 to epoch τ.	$\theta^{(\tau)}$.	$\cos\theta^{(\tau)}$.	$\sin\theta^{(\tau)}$.
0	0	0° 0′	1·000	0·000
1	30	11 27	0·980	0·199
2	61	47 16	0·678	0·735
3	91	58 43	0·519	0·855
4	122	π−85 28	−0·079	0·997
5	152	π−74 1	−0·275	0·961
6	183	π−38 11	−0·786	0·618
7	213	π−26 44	−0·893	0·450
8	243	π−15 18	−0·965	0·264
9	274	π+20 32	−0·937	−0·351
10	304	π+31 59	−0·848	−0·530
11	335	π+67 49	−0·378	−0·926

If $\cos\theta^{(\tau)}$, $\sin\theta^{(\tau)}$ are regarded as quantities having annual and semi-annual inequalities, we may write

$$\cos\theta^{(\tau)} = \alpha_0 + \alpha_1\cos 30^\circ\tau + \beta_1\sin 30^\circ\tau + \alpha_2\cos 60^\circ\tau + \beta_2\sin 60^\circ\tau + \ldots.$$

$$\sin\theta^{(\tau)} = \gamma_0 + \gamma_1\cos 30^\circ\tau + \delta_1\sin 30^\circ\tau + \gamma_2\cos 60^\circ\tau + \delta_2\sin 60^\circ\tau + \ldots.$$

On analysing the numerical values of $\cos\theta^{(\tau)}$, $\sin\theta^{(\tau)}$ by the ordinary processes, I find,

$$\begin{aligned} \alpha_0 &= -0{\cdot}165, & \gamma_0 &= +0{\cdot}273, \\ \alpha_1 &= +0{\cdot}626, & \gamma_1 &= -0{\cdot}500, \\ \beta_1 &= +0{\cdot}756, & \delta_1 &= +0{\cdot}642, \\ \alpha_2 &= +0{\cdot}159, & \gamma_2 &= -0{\cdot}046, \\ \beta_2 &= +0{\cdot}199, & \delta_2 &= +0{\cdot}166. \end{aligned}$$

But in § 2 the harmonic constituents of $\mathfrak{A}_2$ when analysed for

annual and semi-annual inequality were denoted by E_0, E_1, F_1, E_2, F_2, and the constituents of $\mathfrak{B}_2$ were denoted by e_0, e_1, f_1, e_2, f_2. Hence the ten corrections to the E's and F's are (with an easily intelligible alternative notation)

$$\delta E_{0,1,2} = (-P\alpha_{0,1,2} + Q\gamma_{0,1,2}) A_2 + (Q\alpha_{0,1,2} + P\gamma_{0,1,2}) B_2,$$

$$\delta F_{1,2} = (-P\beta_{1,2} + Q\delta_{1,2}) A_2 + (Q\beta_{1,2} + P\delta_{1,2}) B_2,$$

$$\delta e_{0,1,2} = (-R\alpha_{0,1,2} - S\gamma_{0,1,2}) A_2 + (-S\alpha_{0,1,2} + R\gamma_{0,1,2}) B_2,$$

$$\delta f_{1,2} = (-R\beta_{1,2} - S\delta_{1,2}) A_2 + (-S\beta_{1,2} + R\delta_{1,2}) B_2.$$

On substituting the numerical values of α, β, γ, δ, P, Q, R, S, I find

	Coeffit. of A_2.	Coeffit. of B_2.
$\delta E_0 =$	$+0.0029$	$+0.0041$
$\delta e_0 =$	-0.0042	$+0.0031$
$\delta\frac{1}{2}(E_1+f_1) =$	-0.0109	-0.0091
$\delta\frac{1}{2}(E_1-f_1) =$	$+0.0006$	$+0.0020$
$\delta\frac{1}{2}(e_1+F_1) =$	-0.0020	$+0.0000$
$\delta\frac{1}{2}(e_1-F_1) =$	$+0.0091$	-0.0108
$\delta\frac{1}{2}(E_2+f_2) =$	-0.0028	-0.0018
$\delta\frac{1}{2}(E_2-f_2) =$	$+0.0002$	$+0.0012$
$\delta\frac{1}{2}(e_2+F_2) =$	-0.0012	$+0.0001$
$\delta\frac{1}{2}(e_2-F_2) =$	$+0.0017$	-0.0027

Most of these corrections are negligible, but the four which affect the solar elliptic tides T, R must be included, because those tides are so small that a small error affects them sensibly. Hence we may take, with sufficient accuracy,

$$\delta\tfrac{1}{2}(e_1-F_1) = +0.009\,A_2 - 0.011\,B_2, \quad \delta\tfrac{1}{2}(e_1+F_1) = -0.002\,A_2,$$

$$\delta\tfrac{1}{2}(E_1+f_1) = -0.011\,A_2 - 0.009\,B_2, \quad \delta\tfrac{1}{2}(E_1-f_1) = +0.0006\,A_2 + 0.002\,B_2 \qquad \ldots\ldots \quad (16^*),$$

where A_2, B_2 are the components of the M_2 derived from the reduction of that tide by the process of § 6.

Provision for these corrections is made in the computation forms.

§ 4. *Evaluation of* A_0, Sa, S_1, S_2, S_4, S_6, K_2, K_1, P, *when a complete year of observation is not available.*

It is now proposed to consider the case where the period of observation is as much as six complete months and less than a complete year.

The method of the last section apparently depends on the completeness of the year, yet, with certain modifications, it may be rendered available for shorter periods.

We suppose that so much of the year as is available is broken into sets of 30 days by the rules of the last section, and that the means are harmonically analysed. The results of such harmonic analysis for month (τ) are given in (11) of § 3, but for the purpose in hand they now admit of some simplification. It is clear that it is not worth while to evaluate the very small solar elliptic tides T and R from a short period of observation. If, then, we denote by $P^{(\tau)}$ the ratio of the cube of the sun's parallax to its mean parallax at the middle of the month (τ), the first three terms of the third of (11) may be included in the expression $P^{(\tau)} H_s \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} \kappa_s$. The last term of this equation really does involve the solar parallax to some extent, and we may, with sufficient approximation, write the third pair of equations

$$\left.\begin{matrix}\mathfrak{A}_2^{(\tau)} \div P^{(\tau)} \\ \mathfrak{B}_2^{(\tau)} \div P^{(\tau)}\end{matrix}\right\} = H_s \begin{matrix}\cos\\ \sin\end{matrix} \kappa_s + \frac{\mathrm{f}'' H''}{\mathfrak{F}_2} \begin{matrix}\cos\\ \sin\end{matrix} (\kappa'' - \mathrm{V}'' - 60^\circ\tau - 29^\circ{\cdot}53).$$

Let us now consider the value of $P^{(\tau)}$. The longitude of the solar perigee is 281° or −79°, and the ratio of the sun's parallax to its mean parallax is approximately $1 + e_1 \cos(h + 79^\circ)$, and the cube of that ratio is $1 + 3e_1 \cos(h + 79^\circ)$ or $1 + 0{\cdot}0504 \cos(h + 79^\circ)$. Now h, the sun's longitude at the middle of month (τ), is $h_0 + 15^\circ + 30^\circ\tau$; hence

$$P^{(\tau)} = 1 + 0{\cdot}0504 \cos(h_0 + 30^\circ\tau + 94^\circ)$$

and

$$\frac{1}{P^{(\tau)}} = 1 - 0{\cdot}0504 \cos(h_0 + 30^\circ\tau + 94^\circ).$$

Thus it is easy to compute the values of $1/P^{(\tau)}$ for the successive months, when we know h_0 the sun's mean longitude at $0^{\mathrm{d}}\ 0^{\mathrm{h}}$ of the month 0.

The semi-annual tide, being usually small, may be neglected in these incomplete observations, and the equations (11) now become

$$\mathfrak{A}_0^{(\tau)} = A_0 + \frac{H_{sa}}{\mathfrak{F}_1} \cos(\kappa_{sa} - \mathrm{V}_{sa} - 30^\circ\tau - 14^\circ{\cdot}76),$$

$$\left.\begin{matrix}\mathfrak{A}_1^{(\tau)} \\ \mathfrak{B}_1^{(\tau)}\end{matrix}\right\} = H_{\frac{1}{2}s} \begin{matrix}\cos\\ \sin\end{matrix} \kappa_{\frac{1}{2}s} + \frac{\mathrm{f}' H'}{\mathfrak{F}_1} \begin{matrix}\cos\\ \sin\end{matrix} (\kappa' - \mathrm{V}' - 30^\circ\tau - 14^\circ{\cdot}76)$$
$$+ \frac{H_p}{\mathfrak{F}_1} \begin{matrix}\cos\\ \sin\end{matrix} (\kappa_p - \mathrm{V}_p + 30^\circ\tau + 14^\circ{\cdot}76),$$

$$\left.\begin{matrix}\mathfrak{A}_2^{(\tau)} \div \mathrm{P}^{(\tau)} \\ \mathfrak{B}_2^{(\tau)} \div \mathrm{P}^{(\tau)}\end{matrix}\right\} = H_s \begin{matrix}\cos\\ \sin\end{matrix} \kappa_s + \frac{\mathrm{f}'' H''}{\mathfrak{F}_2} \begin{matrix}\cos\\ \sin\end{matrix} (\kappa'' - \mathrm{V}'' - 60^\circ\tau - 29^\circ{\cdot}53),$$

$$\left.\begin{array}{l}\mathfrak{A}_4^{(\tau)}\\ \mathfrak{B}_4^{(\tau)}\end{array}\right\} = H_{2s}\begin{array}{l}\cos\\ \sin\end{array}\kappa_{2s},$$

$$\left.\begin{array}{l}\mathfrak{A}_6^{(\tau)}\\ \mathfrak{B}_6^{(\tau)}\end{array}\right\} = H_{3s}\begin{array}{l}\cos\\ \sin\end{array}\kappa_{3s}, \qquad \frac{1}{P^{(\tau)}} = 1-0{\cdot}0504\cos(h_0+30^\circ\tau+94^\circ) \quad \ldots\ldots \quad (17).$$

When the series of successive values of the $\mathfrak{A}$'s and $\mathfrak{B}$'s are harmonically analysed (by processes which we shall consider shortly) the several coefficients resulting from such analysis will be defined by

$$\mathfrak{A}_0^{(\tau)} = A_0+A_1\cos 30^\circ\tau+B_1\sin 30^\circ\tau,$$

$$\left.\begin{array}{l}\mathfrak{A}_1^{(\tau)}\\ \mathfrak{B}_1^{(\tau)}\end{array}\right\} = \left.\begin{array}{l}\mathrm{C}_0\\ \mathrm{c}_0\end{array}\right\} + \left.\begin{array}{l}\mathrm{C}_1\\ \mathrm{c}_1\end{array}\right\}\cos 30^\circ\tau + \left.\begin{array}{l}\mathrm{D}_1\\ \mathrm{d}_1\end{array}\right\}\sin 30^\circ\tau,$$

$$\left.\begin{array}{l}\mathfrak{A}_2^{(\tau)}\div P^{(\tau)}\\ \mathfrak{B}_2^{(\tau)}\div P^{(\tau)}\end{array}\right\} = \left.\begin{array}{l}\mathrm{E}_0\\ \mathrm{e}_0\end{array}\right\} + \left.\begin{array}{l}\mathrm{E}_2\\ \mathrm{e}_2\end{array}\right\}\cos 60^\circ\tau + \left.\begin{array}{l}\mathrm{F}_2\\ \mathrm{f}_2\end{array}\right\}\sin 60^\circ\tau,$$

$$\text{Mean } \mathfrak{A}_4^{(\tau)} = A_4, \qquad \text{Mean } \mathfrak{B}_4^{(\tau)} = B_4,$$

$$\text{Mean } \mathfrak{A}_6^{(\tau)} = A_6, \qquad \text{Mean } \mathfrak{B}_6^{(\tau)} = B_6 \quad \ldots. \quad (18).$$

Then the subsequent procedure as given in (13) and (14) holds good, the only difference being that we do not obtain the semi-annual and solar elliptic tides.

We shall now consider the harmonic analysis of an imperfect series of values.

It must be premised that each monthly value of $\mathfrak{A}_2^{(\tau)}$, $\mathfrak{B}_2^{(\tau)}$ is to be divided by its corresponding $P^{(\tau)}$ before the analysis is made.

Suppose that $C^{(\tau)}$ denotes a function which is subject to semi-annual inequality, and that

$$C^{(\tau)} = \mathrm{A}_0+\mathrm{A}_2\cos 60^\circ\tau+\mathrm{B}_2\sin 60^\circ\tau.$$

Then it is clear that

$$C^{(0)} = \mathrm{A}_0+\mathrm{A}_2,$$

$$C^{(1)} = \mathrm{A}_0+\tfrac{1}{2}\mathrm{A}_2+\tfrac{1}{2}\sqrt{3}\,\mathrm{B}_2,$$

$$C^{(2)} = \mathrm{A}_0-\tfrac{1}{2}\mathrm{A}_2+\tfrac{1}{2}\sqrt{3}\,\mathrm{B}_2,$$

&c. &c.

I now define D_0, D_1, D_2 thus :—

$$D_0 = C^{(0)}+C^{(1)}+C^{(2)}+\ldots.,$$

$$D_1 = C^{(0)}+\tfrac{1}{2}C^{(1)}-\tfrac{1}{2}C^{(2)}\ldots.,$$

$$D_2 = 0\,.\,C^{(0)}+\tfrac{1}{2}\sqrt{3}\,C^{(1)}+\tfrac{1}{2}\sqrt{3}\,C^{(2)}\ldots.$$

If there be n equations and if they be treated by the method of least squares, we get

$$D_0 = nA_0 + A_2(1+\tfrac{1}{2}-\tfrac{1}{2}\ldots) + B_2(0+\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\sqrt{3}\ldots),$$

$$D_1 = A_0(1+\tfrac{1}{2}-\tfrac{1}{2}\ldots) + A_2(1+\tfrac{1}{4}+\tfrac{1}{4}\ldots) + B_0(0+\tfrac{1}{4}\sqrt{3}-\tfrac{1}{4}\sqrt{3}\ldots),$$

$$D_2 = A_0(0+\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\sqrt{3}\ldots) + A_2(0+\tfrac{1}{4}\sqrt{3}-\tfrac{1}{4}\sqrt{3}\ldots)$$
$$+ B_0(0+\tfrac{3}{4}+\tfrac{3}{4}\ldots).$$

These are the three equations from which A_0, A_2, B_2 are to be found.

A schedule is given below for the formation of D_0, D_1, D_2, and a table of the solutions of these equations according to the number of months available.

Next, suppose $C^{(\tau)}$ denotes a function which is subject to annual inequality, and that

$$C^{(\tau)} = A_0 + A_1 \cos 30^\circ\tau + B_1 \sin 30^\circ\tau.$$

Then
$$C^{(0)} = A_0 + A_1,$$
$$C^{(1)} = A_0 + \tfrac{1}{2}\sqrt{3}A_1 + \tfrac{1}{2}B_1,$$
$$C^{(2)} = A_0 + \tfrac{1}{2}A_1 + \tfrac{1}{2}\sqrt{3}B_1,$$
&c., &c.

In this case the method of least squares gives

$$D_0 = C^{(0)} + C^{(1)} + C^{(2)}\ldots$$
$$= nA_0 + A_1(1+\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\ldots) + B_1(0+\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\ldots),$$

$$D_1 = C^{(0)} + \tfrac{1}{2}\sqrt{3}\,C^{(1)} + \tfrac{1}{2}C^{(2)}\ldots$$
$$= A_0(1+\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\ldots) + A_1(1+\tfrac{3}{4}+\tfrac{1}{4}\ldots) + B_1(0+\tfrac{1}{4}\sqrt{3}+\tfrac{1}{4}\sqrt{3}\ldots),$$

$$D_2 = 0.C^{(0)} + \tfrac{1}{2}C^{(1)} + \tfrac{1}{2}\sqrt{3}\,C^{(1)}\ldots$$
$$= A_0(0+\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\ldots) + A_1(0+\tfrac{1}{4}\sqrt{3}+\tfrac{1}{4}\sqrt{3}\ldots) + B_1(0+\tfrac{1}{4}+\tfrac{3}{4}\ldots).$$

Tables are given below for the formation of D_0, D_1, D_2, and of the solutions of the equations according to the number of months available.

Reduction of incomplete series of results.

Form for finding D_0, D_1, D_2 where the monthly values are subject to semi-annual inequality.

	i.		ii.	iii.	iv.	v.	Factor M.	vi.		
Monthly values.				i + ii.	Last 2 of iii revd.	iii + iv.		$M \times$ v.	iii − iv.	
0	a	6	a'	$a+a'$	....	$a+a'$	1	$a+a'$	....	
1	b	7	b'	$b+b'$	$f+f'$	$b+\ldots+f'$	$\frac{1}{2}$	$\frac{1}{2}(b+\ldots+f')$	$b+\ldots-f'$	$S_4 = 0{\cdot}866$
2	c	8	c'	$c+c'$	$e+e'$	$c+\ldots+e'$	$-\frac{1}{2}$	$-\frac{1}{2}(c+\ldots+e')$	$c+\ldots-e'$	
3	d	9	d'	$d+d'$	....	$d+d'$	-1	$-(d+d')$	....	
4	e	10	e'	$e+e'$						
5	f	11	f'	$f+f'$						
						Sum D_0		Sum D_1	Sum $\times S_4$	
									D_2	

Form for finding D_0, D_1, D_2 where the monthly values are subject to annual inequality.

	i.		ii.	iii.	iv.	v.	vi.	Factor M.	vii.	viii.	Factor M.		
Monthly values.				i + ii.	i − ii.	Last 2 of iv revd.	iv − v.		$M \times$ vi.	iv + v.		$M +$ viii.	
0	a	6	a'	$a+a'$	$a-a'$	....	$a-a'$	1	$a-a'$	$a-a'$	0	0	
1	b	7	b'	$b+b'$	$b-b'$	$f-f'$	$b-\ldots+f'$	S_4	$S_4(b-\ldots+f')$	$b-\ldots-f'$	$\frac{1}{2}$	$\frac{1}{2}(b-\ldots-f')$	
2	c	8	c'	$c+c'$	$c-c'$	$e-e'$	$c-\ldots+e'$	$\frac{1}{2}$	$\frac{1}{2}(c-\ldots+e')$	$c-\ldots-e'$	S_4	$S_4(c-\ldots-e')$	$S_4 = 0{\cdot}866$
3	d	9	d'	$d+d'$	$d-d'$	....	$d-d'$	0	0	$d-d'$	1	$d-d'$	
4	e	10	e'	$e+e'$	$e-e'$								
5	f	11	f'	$f+f'$	$f-f'$								
				Sum D_0					Sum D_1			Sum D_2	

In both these forms those monthly values which are unknown are to be treated as being zero. For example, if 9 months of observation are available, d', e', f' will be zero.

Rule for finding semi-annual inequality from an incomplete series.

Number of months available.		Coefft. of D_0.	Coefft. of D_1.	Coefft. of D_2.
6	A_0 =	+0·167		
	A_2 =		+0·333	
	B_2 =			+0·333
7	A_0 =	+0·148	−0·037	
	A_2 =	−0·037	+0·259	
	B_2 =			+0·333
8	A_0 =	+0·136	−0·045	−0·026
	A_2 =	−0·045	+0·253	−0·019
	B_2 =	−0·026	−0·019	+0·275
9	A_0 =	+0·123	−0·027	−0·047
	A_2 =	−0·027	+0·228	+0·011
	B_2 =	−0·047	+0·011	+0·241
10	A_0 =	+0·107		−0·041
	A_2 =		+0·182	
	B_2 =	−0·041		+0·238

Rule for finding annual inequality from an incomplete series.

Number of months available.		Coefft. of D_0.	Coefft. of D_1.	Coefft. of D_2.
6	A_0 =	+0·977	−0·326	−1·215
	A_1 =	−0·326	+0·442	+0·405
	B_1 =	−1·215	+0·405	+1·845
7	A_0 =	+0·424		−0·528
	A_1 =	+0·250		
	B_1 =	−0·528		+0·990
8	A_0 =	+0·226	+0·062	−0·233
	A_1 =	+0·062	+0·230	−0·093
	B_1 =	−0·233	−0·093	+0·552
9	A_0 =	+0·146	+0·057	−0·098
	A_1 =	+0·057	+0·230	−0·083
	B_1 =	−0·098	−0·083	+0·326
10	A_0 =	+0·110	+0·036	−0·036
	A_1 =	+0·036	+0·218	−0·048
	B_1 =	−0·036	−0·048	+0·218

We thus get the following rule for the evaluation of A_0, S_{sa}, S_1, S_2, S_4, S_6, K_2, K_1, P from 6, 7, 8, 9, or 10 months of observation:—

Proceed as though the year were complete and find the $\mathfrak{A}$*'s and* $\mathfrak{B}$*'s for as many months as are available. Reduce the* $\mathfrak{A}_2$, $\mathfrak{B}_2$ *by multiplication by* $1/P^{(\tau)}$ *or* $1-0\cdot0504\cos(h_0+30°\tau+94°)$.

Analyse $\mathfrak{A}_0^{(\tau)}$, $\mathfrak{A}_1^{(\tau)}$, $\mathfrak{B}_1^{(\tau)}$ *for annual inequality, and* $\mathfrak{A}_2/P^{(\tau)}$, $\mathfrak{B}_2^{(\tau)}/P^{(\tau)}$ *for semi-annual inequality according to the rules for reduction of incomplete series just given.*

Complete the reduction as in § 3.

These rules for reduction do not include the case of 11 months, nor the case where any month in the series is incomplete (*e.g.*, if a fortnight's observation were wanting in one of the months), because these cases may be treated thus:—the $\mathfrak{A}$'s and $\mathfrak{B}$'s return to the same value at the end of a year, and therefore the case of eleven months is the same as that of a missing month at any other part of the year. In both these cases we may interpolate the missing $\mathfrak{A}$'s and $\mathfrak{B}$'s and treat the year as complete.

If three or more weeks of observation were missing they might fall so as to spoil two months, and in this case we should have an incomplete series. It is then to be recommended that the equations of least squares be formed and the equations solved. So many similar cases may arise that it does not seem worth while to solve the equations until the case arises.

§ 5. *Evaluation of* A_0, S_2, S_4, K_2, K_1, P *from a short period of observation.*

If the available tidal observations only extend over a few months, it is useless to attempt the independent evaluation of those tides which we have hitherto found by means of annual and semi-annual inequalities in the monthly harmonic constants. We will suppose that 30 days of observations are available. Then when we neglect the annual tide, and the solar (meteorological) tide S_1, we have from (11) or (17), which give the analysis of 30 days,

$$\mathfrak{A}_0 = A_0,$$

$$\left.\begin{matrix}\mathfrak{A}_1\\ \mathfrak{B}_1\end{matrix}\right\} = \frac{f'H'}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa'-V'-14°\cdot76)+\frac{H_p}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa_p-V_p+14°\cdot76),$$

$$\left.\begin{matrix}\mathfrak{A}_2\\ \mathfrak{B}_2\end{matrix}\right\} = PH_s\begin{matrix}\cos\\ \sin\end{matrix}\kappa_s+\frac{f''H''}{\mathfrak{F}_2}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa''-V''-29°\cdot53),$$

$$\left.\begin{matrix}\mathfrak{A}_4\\ \mathfrak{B}_4\end{matrix}\right\} = H_{2s}\begin{matrix}\cos\\ \sin\end{matrix}\kappa_{2s}, \qquad P = 1+0\cdot0504\cos(h_0+15°).$$

It is now necessary to assume that the P tide has the same

amount of retardation as the K_1, and that the ratio of their amplitudes is the same as in the equilibrium theory. We also make the like assumption with respect to the K_2 and S_2 tides.

Accordingly we put

$$H_p = \tfrac{1}{3}H', \qquad \kappa_p = \kappa'; \qquad H'' = \tfrac{3}{11}H_s, \qquad \kappa'' = \kappa_s.$$

Now since

$$V' = h_0 - \tfrac{1}{2}\pi - \nu', \qquad V_p = -h_0 + \tfrac{1}{2}\pi, \qquad V'' = 2h_0 - 2\nu'',$$

we have

$$\kappa_p - V_p + 14^\circ{\cdot}76 = \kappa' - V' - 14^\circ{\cdot}76 + (2h_0 - \nu' + 29^\circ{\cdot}53) + \pi,$$

$$\kappa'' - V'' - 29^\circ{\cdot}53 = \kappa_s - (2h_0 - 2\nu'' + 29^\circ{\cdot}53).$$

Therefore

$$\left.\begin{matrix}\mathfrak{A}_1\\ \mathfrak{B}_1\end{matrix}\right\} = \frac{f'H'}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa' - V' - 14^\circ{\cdot}76)$$

$$-\frac{\frac{1}{3}H'}{\mathfrak{F}_1}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa' - V' - 14^\circ{\cdot}76 + 2h_0 - \nu' + 29^\circ{\cdot}53),$$

$$\left.\begin{matrix}\mathfrak{A}_2\\ \mathfrak{B}_2\end{matrix}\right\} = PH_s\begin{matrix}\cos\\ \sin\end{matrix}\kappa_s + \frac{\frac{3}{11}f''H_s}{\mathfrak{F}_2}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa_s - 2h_0 + 2\nu'' - 29^\circ{\cdot}53).$$

Let us put $\tan\phi = \dfrac{\sin(2h_0 - \nu' + 29^\circ{\cdot}53)}{3f' - \cos(2h_0 - \nu' + 29^\circ{\cdot}53)}$,

$$\tan\psi = \frac{f''\sin(2h_0 - 2\nu'' + 29^\circ{\cdot}53)}{\frac{11}{3}P\mathfrak{F}_2 + f''\cos(2h_0 - 2\nu'' + 29^\circ{\cdot}53)} \quad \ldots\ldots \quad (19).$$

Then

$$\left.\begin{matrix}\mathfrak{A}_1\\ \mathfrak{B}_1\end{matrix}\right\} = \frac{H'}{\mathfrak{F}'}\frac{[3f' - \cos(2h_0 - \nu' + 29^\circ{\cdot}53]}{3\cos\phi}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa' - V' - 14^\circ{\cdot}76 - \phi),$$

$$\left.\begin{matrix}\mathfrak{A}_2\\ \mathfrak{B}_2\end{matrix}\right\} = H_s\frac{\frac{11}{3}P\mathfrak{F}_2 + f''\cos(2h_0 - 2\nu'' + 29^\circ{\cdot}53)}{\frac{11}{3}\mathfrak{F}_2\cos\psi}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa_s - \psi) \quad .. \quad (20).$$

If therefore

$$\left.\begin{matrix}\mathfrak{A}_1\\ \mathfrak{B}_1\end{matrix}\right\} = R_1\begin{matrix}\cos\\ \sin\end{matrix}\zeta_1, \qquad \left.\begin{matrix}\mathfrak{A}_2\\ \mathfrak{B}_2\end{matrix}\right\} = R_2\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\zeta_2,$$

we have

$$\kappa' = \zeta_1 + V' + 14^\circ{\cdot}76 + \phi = \kappa_p,$$

$$H' = \frac{3\mathfrak{F}_1 R_1\cos\phi}{3f' - \cos(2h_0 - \nu' + 29^\circ{\cdot}53)}, \qquad H_p = \tfrac{1}{3}H'.$$

$$\kappa_s = \zeta_2 + \psi = \kappa'',$$

$$H_s = \frac{\frac{11}{3}\mathfrak{J}_2 R_2 \cos\psi}{\frac{11}{3}P\mathfrak{J}_2 + f'' \cos(2h_0 - 2\nu'' + 29°\cdot 53)}, \quad H'' = \tfrac{3}{11}H_s \,..\, (21).$$

If there be several months available it is recommended that each 30 days be treated quite independently, so that from each group of days we shall get H', κ' and H_s, κ_s. Then the mean value of $H' \cos\kappa'$ is to be taken as the final value of that function, and $H' \sin\kappa'$ is to be treated similarly; finally H', κ' are to be found. The several values of H_s, κ_s may be treated in the same way. Of course we assume throughout that $\kappa_p = \kappa'$, $H_p = \frac{1}{3}H'$, $\kappa'' = \kappa_s$, $H'' = \frac{3}{11}H_s$, assumptions which are usually nearly correct.

The mean value of $\mathfrak{A}_0$ must be taken as giving A_0, but at places with a considerable annual tide it is impossible to obtain a good value of mean water mark from a short series of observations.

§ 6. *On the evaluation of the several tides by grouping of mean solar days.*

Let $n(\gamma-\chi)$ denote the speed in degrees per m.s. hour of any one tide, n being equal to 1, 2, 3, 4, 5, or 6. Then $15°/(\gamma-\chi)$ may be called one "special hour." Since $15°/(\gamma-\eta)$ is one m.s. hour, the ratio of the m.s. to the special hour is $(\gamma-\chi)/(\gamma-\eta)$.

Let one m.s. hour be equal to $1-\beta$ special hour, then

$$\beta = 1 - \frac{\gamma-\chi}{\gamma-\eta}, \text{ special hours.}$$

Let it be required to express the 12^h of any m.s. day of a series of days by reference to special time. It is clear that 12^h m.s. time will be specified by one of the 24 special hours, with something less than half a special hour added or subtracted.

Having fixed the 12^h of m.s. time of a particular m.s. day in the special time scale, let us treat that m.s. day as a whole, and consider the incidence of the other 23 m.s. hours in special time. It is clear that in m.s. time we work backwards and forwards from 12^h by subtracting or adding unity, and that in special time we subtract or add $1-\beta$.

If 12^h m.s. time be $x^h + \alpha$, where α lies between $\pm\frac{1}{2}$ special time, the following is a schedule of equivalence :—

Mean solar time.		Special time.
0^h	=	$(x^h-12^h)+(\alpha+12\beta)$
1^h	=	$(x^h-11^h)+(\alpha+11\beta)$
2^h	=	$(x^h-10^h)+(\alpha+10\beta)$
.....		.....
11^h	=	$(x^h-1^h)+(\alpha+\beta)$
12^h	=	$x^h+\alpha$
13^h	=	$(x^h+1^h)+(\alpha-\beta)$
.....		.....
22^h	=	$(x^h+10^h)+(\alpha-10\beta)$
23^h	=	$(x^h+11^h)+(\alpha-11\beta)$

In the column of special time it is supposed that 24^h is added or subtracted, so that the result is less than 24^h. For example, if x is 10, the hour column of special time will run 22^h, 23^h, 0^h, ..., 9^h, 10^h, 11^h, ..., 20^h, 21^h.

If the series of days be long x will have all integral values between 0 and 23 with equal frequency, and since α has all values between $+\frac{1}{2}$ and $-\frac{1}{2}$ with equal frequency, the excess of the solar hour above the nearest exact special hour (which may be called the error) will have all its possible values with equal frequency. If the mean solar hours be arranged in a schedule of columns headed 0^h, 1^h, ..., 23^h of special time, each column will be subject to errors which follow the same law of frequency.

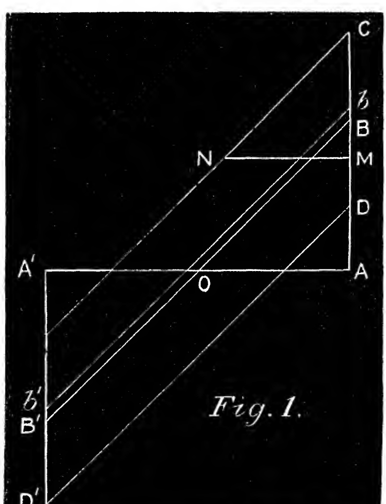


Fig. 1.

Let abscissæ (fig. 1) measured from O along A′OA represent magnitude of α.

Since α lies between $\pm\frac{1}{2}$, the limit of the figure is given by $OA = OA' = \frac{1}{2}$.

If magnitude of error (*i.e.* m.s. — special hour), measured in special time, be represented by ordinates, a line BOB′ at 45° to AOA′ represents all the errors which can arise in the incidence of the m.s. 12^h in the schedule of special time.

If a line bb' be drawn parallel to and above BB′ by a distance β, we have a representation of all the errors of incidence of the m.s. 11^h. If a series of equidistant parallel lines be drawn above and below BB′ until there are 12 above and 11 below, then the errors of all the m.s. hours are represented, the top one showing the errors of the m.s. 0^h and the bottom one the errors of the m.s. 23^h.

Any special hour corresponds with equal frequency with each solar hour, and hence each mode of error occurs with equal frequency.

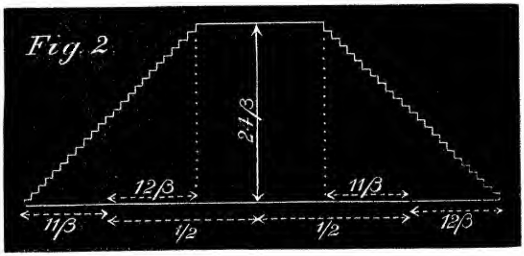
It is now necessary to consider in how many ways an error of given magnitude can occur. If in the figure AM represents an error of given magnitude, then wherever MN cuts a diagonal line, it shows that an error may arise in one way.

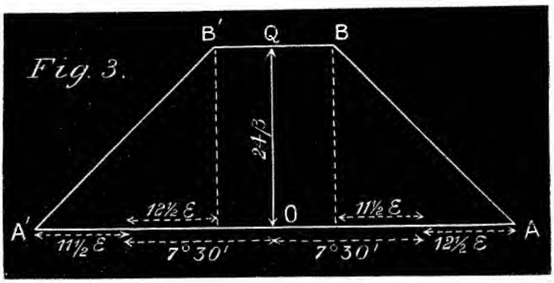
It is thus clear that there are no + errors greater than $\frac{1}{2}+12\beta$, and no — errors greater than $\frac{1}{2}+11\beta$, and

Errors of magnitude.		
$\frac{1}{2}+12\beta$ to $\frac{1}{2}+11\beta$	may arise in	1 way.
$\frac{1}{2}+11\beta$ to $\frac{1}{2}+10\beta$	,,	2 ways.
$\frac{1}{2}+10\beta$ to $\frac{1}{2}+9\beta$	,,	3 ways.
....		
$\frac{1}{2}-10\beta$ to $\frac{1}{2}-11\beta$	,,	23 ways.
$\frac{1}{2}-11\beta$ to $-(\frac{1}{2}-12\beta)$	,,	24 ways.
$-(\frac{1}{2}-12\beta)$ to $-(\frac{1}{2}-11\beta)$	,,	23 ways.
....		
$-(\frac{1}{2}+9\beta)$ to $-(\frac{1}{2}+10\beta)$	,,	2 ways.
$-(\frac{1}{2}+10\beta)$ to $-(\frac{1}{2}+11\beta)$	,,	1 way.

The frequency of error is represented graphically in fig. 2. The slope of the two staircases is drawn at 45°, but any other slope would have done equally well.

A frequency curve of this form is not very convenient, and, as there are many steps in the ascending and descending slopes, I substitute the frequency curve shown in fig. 3. This is clearly equivalent to the former one. In fig. 3 all the times shown in fig. 2 are converted to angle at 15° to the hour; ϵ accordingly denotes $15°\beta$.





Now let $\cos n\,(\theta - x)$ be the observed value of a function whose true value is $\cos n\,\theta$, and suppose that x, the error of θ, has a frequency $f(x)$; then the mean value of the function deduced from many observations will be

$$\int_{-\infty}^{+\infty} f(x) \cos n\,(\theta - x)\,dx \div \int_{-\infty}^{+\infty} f(x)\,dx.$$

In our case $f(x)$ is the ordinate of the frequency curve whose abscissa is x.

Let $OQ = h$, $QB = a$, $QB' = b$, $OA = a+h$, $OA' = b+h$; then

$$\int_{-\infty}^{+\infty} f(x)\,dx = (a+b+h)\,h.$$

$$\int_{-\infty}^{+\infty} f(x) \cos n\,(\theta - x)\,dx$$

$$= \int_0^a h \cos n\,(\theta - x)\,dx + \int_a^{a+h} (a+h-x) \cos n\,(\theta - x)\,dx$$

$$+ \int_0^b h \cos n\,(\theta + x)\,dx + \int_b^{b+h} (b+h-x) \cos n\,(\theta + x)\,dx,$$

$$= \frac{4}{n^2} \cos n\left[\theta - \tfrac{1}{2}(a-b)\right] \sin \tfrac{1}{2} nh \sin \tfrac{1}{2} n\,(a+b+h).$$

The algebraical steps involved in the evaluation of these four integrals and subsequent simplification are omitted.

Hence the result is

$$\frac{\sin\frac{1}{2}nh}{\frac{1}{2}nh}\,\frac{\sin\frac{1}{2}n(a+b+h)}{\frac{1}{2}n(a+b+h)}\cos n\left(\theta-\tfrac{1}{2}(a-b)\right).$$

By reference to the figure it is clear that

$$a+b+h=15^\circ,\quad h=24\epsilon,\quad a=7\tfrac{1}{2}^\circ-11\tfrac{1}{2}\epsilon,\quad b=7\tfrac{1}{2}^\circ-12\tfrac{1}{2}\epsilon,\quad a-b=\epsilon.$$

Write, then,

$$\mathfrak{F}_n=\frac{12\,n\epsilon}{\sin 12\,n\epsilon}\;\frac{\frac{15}{2}n}{\sin\frac{15}{2}n},$$

and we obtain as the mean value of $\cos n\theta$, when found in this way,

$$\frac{1}{\mathfrak{F}_n}\cos n\left(\theta-\tfrac{1}{2}\epsilon\right).$$

It is obvious that if we had begun with $\sin n\theta$, the argument in the result and the factor $\mathfrak{F}_n$ would have been the same. Accordingly, a function $R'\cos(n\theta-\zeta')$ would yield the result $\dfrac{R'}{\mathfrak{F}_n}\cos\left[n\left(\theta-\tfrac{1}{2}\epsilon\right)-\zeta'\right]$. If 24 equidistant results of this sort are submitted to harmonic analysis to find A_n, B_n, we shall get—

$$A_n=\frac{R'}{\mathfrak{F}_n}\cos\left(\zeta'+\tfrac{1}{2}n\epsilon\right)=R\cos\zeta,\text{ suppose,}$$

$$B_n=\frac{R'}{\mathfrak{F}_n}\sin\left(\zeta'+\tfrac{1}{2}n\epsilon\right)=R\sin\zeta,\text{ suppose.}$$

Accordingly $\quad R=\dfrac{R'}{\mathfrak{F}_n},\qquad \zeta=\zeta'+\tfrac{1}{2}n\epsilon.$

But it is required to find R', ζ', so that

$$R'=\mathfrak{F}_nR,\qquad \zeta'=\zeta-\tfrac{1}{2}n\epsilon.$$

Thus when the 24 observed hourly tide heights on any m.s. day are regrouped so that the observed height at 12^{h} m.s. time is reputed to appertain to an exact special hour, and each of the previous and subsequent hourly values of that m.s. day are reputed to belong to previous and subsequent exact special hours; and when a long series of m.s. days are treated similarly, and when the mean heights of water at each of the 24 special hours are harmonically analysed, we shall obtain the required result by augmenting R by a factor $\mathfrak{F}_n$, and by subtracting $\frac{1}{2}n\epsilon$ from ζ.

The values of $\mathfrak{f}_n$ and of $\frac{1}{2}n\epsilon$ will be different for each kind of tide, and the following table gives their numerical values.

Table of $\mathfrak{f}_n$ and $\frac{1}{2}n\epsilon$.

Initial of tide.	n.	$\log \mathfrak{f}_n$.	$\frac{1}{2}n\epsilon$.
M_1	1	0·00212	0°·26
M_2	2	0·00849	0°·53
M_3	3	0·01915	0°·79
M_4	4	0·03416	1°·05
M_6	6	0·07767	1°·57
N	2	0·01361	0°·82
L	2	0·00570	0°·24
ν	2	0·01278	0°·78
O	1	0·00535	0°·57
J	1	0·00225	−0°·28
Q	1	0·01149	0°·90
μ	2	0·02016	1°·09
2SM	2	0·00805	−0°·49
MS	4	0·02342	0°·52
λ	2	0·00595	0°·28
2N	2	0·02136	1°·13
OO	1	0·00481	−0°·53
MK	3	0·01438	0°·50
2MK	3	0·02632	1°·09
MN	4	0·04328	1°·35

As it does not appear worth while to evaluate the tides written below the line, no use will be made of the last six results given in this table.

§ 7. *On the periods over which the means are to be taken in evaluating the tidal constants.*

We have considered in previous sections the treatment of the group of tides which are associated with solar time, when the period of observation is less than a year, and we have now to consider the other tides.

It is important that the means be taken over such a number of days that the perturbation arising from other tides shall be minimised.

The perturbation between semi-diurnal and diurnal tides is always negligible. It is therefore only necessary to consider the action of the tides M_2, S_2 in the case of semi-diurnal tides, and that of K_1 and O for diurnal tides.

It is easy to see that the influence of a disturbing tide is evanescent when the means are taken over a period such that the excess of the argument of the disturbed over that of the disturbing tide has increased through a multiple of 360°. As, however, we are working with integral numbers of days, and as the speeds of tides are incommensurable, this condition cannot be exactly satisfied.

From this consideration it appears that to minimise the perturbation of S_2, 2SM, μ by M_2 (and *vice versâ*) we must stop at an exact multiple of a semi-lunation. To minimise the effect of M_2 on N and L, and of K_1 on J and Q, we must stop at an exact multiple of a lunar anomalistic period. To minimise the effect of M_2 on ν, we must stop at a multiple of the period $2\pi/(\sigma+\varpi-2\eta)$. To minimise the effect of K_1 on O, we must stop at an exact multiple of a semi-lunar period.

For the quater-diurnal tide, MS, it is immaterial where we stop, and so it may as well be taken at a multiple of a semi-lunation.

The following table (p. 374) gives the rules derived from these considerations.

§ 8. *On the tides of long period.*

The annual (Sa) and semi-annual (Ssa) tides are evaluated in the course of the work by which other important tides are found. These are the only two tides of long period which have a practical importance in respect to tidal prediction, but the luni-solar fortnightly (MSf), the lunar fortnightly (Mf), and the lunar monthly (Mm) tides have a theoretical interest.

It will therefore be well to show how they may be found. The process is short, and, although it is less accurate than the laborious plan followed in the Indian reductions, it appears to give fairly good results.

For the sake of simplicity, let us consider the tide MSf. Its period is about 14 days, and therefore a day does not differ very largely from a twelfth part of the period. Accordingly, if about two days in a fortnight are rejected by proper rules, the mean heights of water on the remaining days may be taken as representatives of twelve equidistant values of water height.

I therefore go through the whole year and reject, according to proper rules, the daily sums of the 24 hourly heights corresponding to certain 69 of the days out of 369. The remaining 300 values are written consecutively into a schedule of 12 columns and 25 rows, of which each corresponds to a half lunation. The 12 columns are summed, and the sums are harmonically analysed for the first pair of harmonic components. These components have to be divided by 24 times 25, or by 600, because the daily mean water height is $\frac{1}{24}$th of the daily sum, and there are 25 semi-lunations.

In the same way the semi-lunar period is about $13\frac{1}{2}$ days, and if

Number of the last day to be included in the evaluation of the several tides for observations extending over any period up to a year.

For M_2, μ, 2SM, MS. Stop with one of the following days (semi-lunations).	For O. Stop with one of the following days (semi-lunar periods).	For N, L, J, Q. Stop with one of the following days (anom. periods).	For ν. Stop with one of the following days (periods $2\pi/(\sigma+\varpi-2\eta)$).
14	13	27	31
29	26	54	63
43	40	——	——
58	54	74+ 8	74+20
73	67	+35	+52
——	——	+63	——
74+14	74+ 7	——	148+10
+28	+21	148+16	+42
+43	+34	+44	——
+58	+48	+71	222+ 0
+73	+62	——	+31
——	——	222+25	+63
148+13	148+ 1	+53	——
+28	+15	——	296+21
+43	+29	296+ 6	+53
+58	+42	+34	——
+72	+56	+61	
——	+70	——	
222+13	——		
+28	222+ 9		
+43	+23		
+58	+37		
+72	+50		
——	+64		
296+13	——		
+28	296+ 4		
+43	+17		
+57	+32		
+72	+45		
——	+58		
	+72		
	——		

we erase by proper rules 45 daily sums out of 369, we are left with 324, which may be written consecutively in a schedule of 12 columns and 27 rows, of which each corresponds to a semi-lunar period. The summing and analysis is the same as in the last case, but the final division is by 24 times 27, or by 648.

In this way we evaluate the luni-solar fortnightly and lunar fortnightly inequalities in the height of the water.

The period of the moon is between 27 and 28 days, and if we erase appropriately about one day in eight we are left with sets of 24 values which may be taken as 24 equidistant values of the daily sums. Accordingly we erase 46 daily sums out of 358, and write the 312 which remain consecutively into a schedule of 24 columns and 13 rows, of which each corresponds to a lunar anomalistic period.

The 24 columns are summed and the sums analysed for the first components. Finally, the components are to be divided by 24 times 13, or by 312. In this way the lunar monthly tide is evaluated.

But the result obtained in this way is, as far as concerns the tide MSf, to some, and it may be to a large, extent fictitious. It represents, in fact, a residuum of the principal lunar tide M_2. That this is the case will now be proved.

Suppose that t_0 is an integral number of days since epoch, being the time of noon on a certain day; then the principal lunar tide M_2 on that day may be written $H_m \cos[2(\gamma-\sigma)(t_0+\tau)-\zeta_m]$, where τ is less than 24 hours. Then the daily sum for that day will be

$$H_m \frac{\sin 24(\gamma-\sigma)}{\sin(\gamma-\sigma)} \cos[2(\gamma-\sigma)\,t_0+23(\gamma-\sigma)-\zeta_m].$$

Now since t_0 is an integral number of days $2(\gamma-\sigma)t_0$ only differs from $-2(\sigma-\eta)\,t_0$ by an exact multiple of 360°; hence the argument of the cosine may be written $2(\sigma-\eta)\,t_0-23(\gamma-\sigma)+\zeta_m$.

But the true luni-solar fortnightly tide, which we may denote $\mathrm{f}H\cos[2(\sigma-\eta)(t_0+\tau)-\zeta]$, varies so slowly in the course of a day that the daily sum is sensibly equal to

$$24\,\mathrm{f}H\cos[2(\sigma-\eta)\,t_0+23(\sigma-\eta)-\zeta].$$

It thus appears that the residual effect of M_2 is of exactly the same form as that of MSf. It becomes, therefore, necessary to clear the harmonic components, determined as described above, from the effects of M_2.

In order to determine the values of these clearances, I found the values of $\cos 2(\sigma-\eta)t$ and $\sin 2(\sigma-\eta)t$ for every noon in a year of 369 days. I then erased the values selected for the treatment of MSf and analysed the remaining values. In this way it was easy to find the effect of the known M_2 tide.

Suppose that A_1, B_1 are the first harmonic components determined by the treatment of a series of daily sums, and that δA_1, δB_1 are the corrections to be applied to them to eliminate the effects of M_2, then I find that if A_m, B_m are the two components of M_2 as determined by the previous method (§ 6) of analysis,

$$\delta A_1 = +0{\cdot}0304\, A_m - 0{\cdot}0171\, B_m,$$

$$\delta B_1 = -0{\cdot}0171\, A_m - 0{\cdot}0304\, B_m,$$

$$C = A_1 + \delta A_1, \qquad\qquad D = B_1 + \delta B_1.$$

$$C - 0{\cdot}047\, D = 0{\cdot}992\, \mathrm{f} H \cos \zeta,$$

$$D + 0{\cdot}047\, C = 0{\cdot}992\, \mathrm{f} H \sin \zeta.$$

Whence f being known from Baird's manual (being a function of the longitude of moon's node), H and ζ are determinable. We have also

$$\kappa = \zeta + 2\,(s_0 - \xi - h_0 + \nu) + 11^{\circ}{\cdot}7.$$

In the set of computation forms which I have prepared for use on the present plan, it is shown what days are to be erased for each of the three analyses, and how they are to be entered in schedules, summed, and analysed.

§ 9. *On abridgment in the computations.*

It seemed probable that one decimal of a foot would suffice to express the hourly tide heights. In order to test this, I have taken several individual days of observation at Port Blair, and have found, by harmonic analysis, the time and amplitude of the diurnal and semi-diurnal H.W., first, when the hourly heights are expressed to two decimal places of a foot, and secondly, when they are only entered to the nearest tenth of a foot. I find that the times of H.W. agree within less than a minute of time, and that the amplitudes agree within a fraction of an inch. If this much be true of individual days, the difference of results arising from two or one place of decimals will clearly entirely disappear when a series of days is considered. Hence, by taking as unit the tenth of a foot, or the inch, or even two inches at places with large tides, we may always express all, or nearly all, the heights on which we are to operate by two significant figures. The adoption of this rule not only saves the writing of a large number of figures, but also enormously diminishes the labour of the additions which have to be made.

It also seemed probable that substantial accuracy might be attained from the harmonic analysis of only 12 hourly values instead of 24. In order to test this I took the tidal reductions for Port Blair, Andaman Islands (kindly lent me by the Survey of India), and have compared the results which would have been derived from 12 values with those actually obtained from 24 values by the computers of the Indian Survey. The following tables give the results:—

Semi-diurnal tides.

Initial.	Results from 12 two-hourly values.	Results from 24 hourly values.	Error, (12) – (24).
	ft.	ft.	ft.
S $\begin{cases} A_2 = \\ B_2 = \end{cases}$	+0·6883 +0·6775	+0·6890 +0·6768	−0·0007 +0·0008
M $\begin{cases} A_2 = \\ B_2 = \end{cases}$	−1·7032 +1·0883	−1·7005 +1·0872	−0·0027 +0·0011
K $\begin{cases} A_2 = \\ B_2 = \end{cases}$	−0·1437 −0·2527	−0·1407 −0·2515	−0·0030 −0·0012
L $\begin{cases} A_2 = \\ B_2 = \end{cases}$	+0·0357 +0·0612	+0·0347 +0·0610	+0·0010 +0·0002
N $\begin{cases} A_2 = \\ B_2 = \end{cases}$	+0·3422 +0·2192	+0·3486 +0·2124	−0·0064 +0·0068
ν $\begin{cases} A_2 = \\ B_2 = \end{cases}$	−0·1217 +0·0867	−0·1165 +0·0887	−0·0052 −0·0020
μ $\begin{cases} A_2 = \\ B_2 = \end{cases}$	+0·0857 +0·0383	+0·0849 −0·0388	−0·0008 −0·0005
2SM $\begin{cases} A_2 = \\ B_2 = \end{cases}$	+0·0055 −0·0198	+0·0037 −0·0200	+0·0018 +0·0002

Diurnal tides.

Initial.	Result from 12 two-hourly values.	Result from 24 hourly values.	Error, (12) – (24).
	ft.	ft.	ft.
S $\begin{cases} A_1 = \\ B_1 = \end{cases}$	+0·0175 +0·0223	+0·0185 +0·0216	−0·0010 +0·0007
M $\begin{cases} A_1 = \\ B_1 = \end{cases}$	+0·0120 −0·0168	+0·0059 −0·0173	+0·0061 +0·0005
K $\begin{cases} A_1 = \\ B_1 = \end{cases}$	+0·3815 +0·1398	+0·3847 +0·1396	−0·0032 +0·0002
O $\begin{cases} A_1 = \\ B_1 = \end{cases}$	−0·0818 +0·1335	−0·0729 +0·1386	−0·0089 −0·0051
P $\begin{cases} A_1 = \\ B_1 = \end{cases}$	−0·0167 −0·1287	−0·0178 −0·1280	+0·0011 −0·0007
J $\begin{cases} A_1 = \\ B_1 = \end{cases}$	−0·0193 +0·0315	−0·0167 +0·0347	−0·0026 −0·0032
Q $\begin{cases} A_1 = \\ B_1 = \end{cases}$	+0·0140 −0·0170	+0·0136 −0·0194	+0·0004 +0·0024

The mean discrepancy in the case of the semi-diurnal tides is 0·0022 ft., and the greatest is +0·0068; in the case of the diurnal tides the mean discrepancy is 0·0026 ft., and the greatest is 0·0089.

In tidal work results derived from different years of observation differ far more than do these two sets of results, and hence the analysis of 12 two-hourly values for diurnal and semi-diurnal tides gives adequate results.

I find that this abbreviation does not give satisfactory results for quater-diurnal tides, and the sixth harmonic is not derivable from 12 values. Therefore, when these tides are to be evaluated the 24 hourly values must be used.

It will still be necessary to write all the 24 hourly heights on each computing strip, but when the strips are put into any one of the arrangements, except where quater-diurnal tides are required, we need only add up the columns 0, 2, 4, ..., 22, and may omit the columns 1, 3, ..., 23.

§ 10. *On a trial of the proposed method of reduction.*

As already mentioned, I have the tidal reductions for one year (beginning April 19, 1880) for Port Blair, Andaman Islands. I am thus able to make a comparison between the results of the old method and of the new. The computation was, in large part, done for me by Mr. Wright.

It appeared sufficient to evaluate the tides of the S series and those allied with them, the tides of the M series, and the tide Q; also the tides of long period MSf, Mf, Mm.

The S series test the new process of harmonic analysis of monthly harmonic components for annual and semi-annual inequalities. I chose M because it is the most important tide, and Q because it puts the proposed method of grouping to a severe test, and is very small in amplitude.

In the Q time scale the day is $26^h\ 52^m$ of mean solar time, from which it follows that one of the 24 mean solar hourly observations may fall as much as $2^h\ 0^m$ away from the exact Q hour to which it is reputed to belong. Thus the hourly observations are arranged in wide groups round the Q hours, and the hypothesis involved in the method is put to a severe strain.

Lastly, the results for tides of long period test my proposed abridgment.

It will be seen in the table on p. 379 that the two methods give results in close agreement. There is, however, a sensible discrepancy in the K_2 tide, but in this case I am inclined to accept the new value as better than the old one. This tide is governed by sidereal time, which differs but little from mean solar time. Hence, in the Indian

Port Blair. 1880–81.

		I. Indian calculation.	N. New method.	I−N (height).	I−N (phase).
		ft.	ft.	ft.	
A_0	=	4·792	4·795	−0·003	
Sa	H =	·299	·299	000	
	κ =	163°	162°	..	+1°
Ssa	H =	·106	·111	− ·005	
	κ =	165°	164°	..	+1°
T	H =	·099*	·094		
	κ =	313°*	339°		
R	H =	·020*	·004		
	κ =	326°*	312°		
S_1	H =	·028	·026	+ ·002	
	κ =	49°	53°	..	−4°
S_2	H =	·966	·973	− ·007	
	κ =	316°	315°	..	+1°
S_4	H =	·003	·003	000	
	κ =	107°	105°	..	+2°
K_1	H =	·403	·401	+ ·002	
	κ =	326°	326°	..	0°
K_2	H =	·286	·268	+ ·018	
	κ =	314°	311°	..	+3°
P	H =	·130	·139	− ·009	
	κ =	324°	323°	..	+1°
M_1	H =	·014	·013	+ ·001	
	κ =	23°	34°	..	−11°
M_2	H =	2·042	2·043	− ·001	
	κ =	279°	279°	..	0
M_3	H =	·004	·004	000	
	κ =	20°	54°	..	−34°
M_4	H =	·003	·006	− ·003	
	κ =	167°	264°	..	−97°
M_6	H =	·004	·005	− ·001	
	κ =	342°	315°	..	+27°
(24 values) Q	H =	·023	·023	000	
	κ =	236°	233°	..	+ 3°
(12 values) Q	H =	·023	·022	+ ·001	
	κ =	236°	234°	..	+ 2°

* These are derived from 1880–82.

Tides of Long Period.

	I. Indian calculation.	N. New method.	I−N (height).	I−N (phase).
	ft.	ft.	ft.	
MSf $\begin{cases} H = \\ \kappa = \end{cases}$	·045 163°	·019 168°	+ ·026 ..	−5°
Mf $\begin{cases} H = \\ \kappa = \end{cases}$	·056 356°	·056 356°	000 ..	0
Mm $\begin{cases} H = \\ \kappa = \end{cases}$	·016 12°	·020 13°	− ·004 ..	−1°

method of grouping, considerable errors of incidence of the S hours in the K time scale prevail for many days together, and the method seems of doubtful propriety. The same is true of the P tide, and here also the two methods give somewhat different results.

The accuracy with which the very small Q tide comes out, whether from 24 values or only from 12, is surprising, and may perhaps be, to some extent, due to accident. It shows, however, that the present method may be safely applied, even when the special time scale differs considerably from mean solar time.

The results for the tides of long period are quite as close to the old values as could be expected.

§ 11. *A comparison of the work involved in the new and old methods of reduction.*

It has been usual in the Indian reductions to use three digits in expressing the height of water, and there have been 15 series, or even more. Now $3 \times 24 \times 365 \times 15$ is 394000; hence the computer has had to write that number of figures in reducing a year of observation. This does not include the evaluation of the annual and semi-annual tides, so that we may say that there have been about 400,000 figures to write.

I propose to express the heights by two digits, and they only have to be written once. Thus, in the present plan, the number of figures to write is $2 \times 24 \times 365$, or 17,500. Thus the writing of 382,000 figures is saved.

In the old method the computer had to add together all the digits written, say, 394,000 additions of digit to digit.

I propose to use 24 hourly values in three series, viz., S, M, and MS, and 12 two-hourly values in eight others. Therefore, the number of additions will be $3 \times 2 \times 24 \times 365 + 8 \times 2 \times 12 \times 365$ or 123,000. Thus 270,000 additions are saved.

We may say that formerly there were about 800,000 operations (writing and addition), and that in the present method there will be about 140,000. This estimate does not include a saving of several thousands of operations in obtaining the tides of long period. I am therefore within the mark when I claim that the work formerly bestowed on one year of observation will now reduce at least five years.

It has been found that the manufacture of my computing strips of xylonite is rather expensive, but as it formerly cost in England rather more than £20 to reduce a year of observation, the cost of the apparatus will be covered by the saving in the reduction of a single year, and it will serve for any length of time.

§ 12. *On the completion of the record for short gaps and long gaps.*

In any long series of tidal observations there are usually some breaks in the record in consequence of the stoppage of the clock of the tide gauge, or from some other cause. Now the process of elimination by grouping depends essentially on the completeness of the record, and it is therefore necessary to fill in blanks by interpolation.

Such interpolation has not been usual in the operations of the Indian Survey, and it might be thought that the complete omission of the missing entries is the proper course to take; but it is easily shown that this treatment is exactly equivalent to the assumption that the water remained stagnant at mean sea level during the whole time of stoppage of the gauge. It is obvious, therefore, that any conjectural values are better than none.

The process by which it is proposed to interpolate is best shown by an example.

At Port Blair (beginning April 19, 1880) the column of 6^h from days 99 to 112 gives the heights shown in the first column of the table below. I suppose that the tide gauge broke down on day 103, and only came into action again on day 110.* There was really no break down, and the actuality during the supposed hiatus is shown in the last column but one.

Now if we look back about a month we find that the water stood about the same height at the same hour of the day (viz., 6^h). Then the "previous record" (which is complete) beginning at 69^d is entered in the next column. Similarly a "subsequent record" is found about a month later, and is entered in a third column. The mean of the previous and subsequent records is then taken as giving the values to be interpolated.

* The days are here numbered from 1, instead of from 0. This has been the usage in India hitherto.

The last two columns contain a comparison between the interpolation and what in the present case we know to have been actuality. There is a mean error of 0·20 ft. Thus it is clear that a fair record may be interpolated even with so long a break as a week.

In this example I have only shown the interpolation for one column, but of course all the other twenty-three columns would really have to be treated similarly.

I find by trial that the result would be a little improved by a graphical method, but that process is slightly more troublesome than the numerical one.

Table of Interpolation.

Defective record.		Previous record.		Subsequent record.		Mean of previous and subsequent.	Actuality.	Error.
Day.	6^h.	Day.	6^h.	Day.	6^h.			
99	2·54	69	2·02	129	2·56	2·29		
100	3·13	70	2·83	130	3·07	2·95		
101	3·86	71	3·70	131	3·70	3·70		
102	4·40	72	4·55	132	4·27	4·41		
103	..	73	5·10	133	4·88	4·99	4·83	+0·16
104	..	74	5·60	134	5·27	5·44	5·24	+0·20
105	..	75	5·72	135	5·35	5·54	5·39	+0·15
106	..	76	5·67	136	5·28	5·48 Interpolation.	5·18	+0·30
107	..	77	5·59	137	4·92	5·26	4·90	+0·36
108	..	78	5·04	138	4·44	4·74	4·53	+0·21
109	..	79	4·62	139	3·57	4·10	4·05	+0·05
110	3·32	80	3·81	140	2·95	3·38		
111	2·64	81	3·26	141	2·50	2·38		
112	2·17	82	2·73	142	1·99	2·36		

It may happen that the hiatus is too long for treatment in this way. I do not think it would be safe to treat much more than a fortnight by interpolation.

It has been shown in § 4 how the tides associated with S are to be treated where the record is deficient, and it remains to consider the other tides.

In § 7 are given the days with which we must stop in the analysis of an incomplete year, and this table affords us the means of treating a long hiatus in the observation.

We may in fact omit all the entries between any two of the numbers given in the table without seriously affecting the result.

Let us suppose, as an example, that the tide gauge broke down on day 210 and was only repaired and in operation again on day 226. Now 210 is 148+62, and 225 is 222+3.

Then we see by the table in § 7 that in finding the means for

M, 2SM, MS, when the computing strips are written for the third time, we must remove strips 59, 60, 61 (which have numbers written on them) and may leave the remaining strips of that writing which are blank. When the strips are written for the fourth time strips 0, 1, 2, 3 will be blank, but we must remove strips 4 to 13 inclusive. When all the strips are used in a complete year there are 369, and this is the divisor used in obtaining the harmonic constants, but when there is this supposed hiatus we do not use 15 strips of the third writing and 14 strips of the fourth writing, so that the divisor will be 340.

Again, when we are evaluating O in the third writing, strips 57, 58, 59, 60, 61 must be removed, and in the fourth writing strips 4 to 9 inclusive. In a complete year the divisor is 369, but we now do not use 17 strips of the third writing and 10 of the fourth writing, so that the divisor becomes 342.

Again, in evaluating N, L, J, Q, in the third writing we remove strips 45 to 61 inclusive, and in the fourth writing strips 4 to 25 inclusive. The divisor is reduced from 358 to 303.

Lastly in evaluating ν, in the third writing strips 43 to 61 inclusive are removed, and in the fourth writing strips 4 to 31 inclusive. The divisor is reduced from 350 to 287.

Any hiatus, be it long or short, may be treated in this way, but it is clear that if it be short enough to treat by interpolation, it is best to adopt that method.

INSTRUCTIONS FOR USING THE COMPUTING APPARATUS.

The apparatus for the reduction of tidal observations, together with computation forms, can be purchased from the Cambridge Scientific Instrument Company at a price (as far as can be now foreseen) of about £8.

In case of any insufficiency in the following instructions recourse must be taken to the preceding paper.

On the degree of accuracy requisite in the hourly heights.

It will usually be sufficient if the heights be measured to within one-tenth of a foot, and the decimal point may, of course, be omitted in computation.

This gives amply sufficient accuracy at a place where the semi-range of the principal lunar tide is 2 ft., and where spring range is from 6 ft. to 7 ft.

At some places with small tides a smaller unit might be necessary, and at others with very large tides a unit of 2 in., or of a fifth of a foot, might suffice.

Whatever unit of length be taken it is important, for the saving of work, and it is sufficient, that all or nearly all the heights should be expressed by two digits.

Completion of record.

If there is an accidental break in the record, it is very important that it should be completed according to the method shown in § 12, or by some other equivalent plan.

The computation forms are drawn up on the supposition that the year of observation is complete, but with proper alterations, which will now be indicated, they may be used in other cases.

In § 4 it is shown how to treat the tides of the S group when the observations have been subject to a long stoppage in the course of the year, and also when the observations extend over any period from six months to a year.

In § 5 it is shown how to treat the S group for a short period of observation.

If the stoppage be a long one, the method explained in § 12 must be adopted for all the other tides. The same section also shows the treatment for observations extending over any period, long or short, less than a year.

Entries and summations.

The computing strips are intended to take writing in *pencil* or *liquid Indian ink*, but not in common ink.

They are to be cleaned with a damp cloth, and a little soda may be put in the water if they become greasy.

Lay the red S sheet on one drawing board and set up the strips with their ends abutting against the corresponding numbers. The strip numbered 60 is also to be put on the board.

Write the hourly heights for each day on the strip bearing the corresponding number, strip 0 for day 0, strip 1 for day 1, and so on up to strip 73 for day 73. The 24 hourly heights are to be written in the 24 divisions of each strip, beginning on the left with 0^h and ending on the right with 23^h.

Remove strip 60.

Sum the 24 columns formed by the divisional marks on consecutive sets of 30 strips. Thus, days 0 to 29 afford 24 sums; days 30 to 59 afford the second set of 24 sums; days 61 to 73 afford 24 sums, which are the beginning of a third 30, to be completed when the second set of 74 days shall have been written on the strips.

The numbers 0, 1, 2, ..., 23, 0, 1, ..., 23 at the head and foot of the guide sheet indicate the hours corresponding to the columns.

The sums of the columns on the board are to be entered in the

corresponding columns of the form "Hourly sums of S series in twelve months."

Lay the red M guide sheet on the other drawing board, and transfer the strips from the first board to the new arrangement shown by the zigzag lines, strip 60 being now reintroduced.

There will now be 48 columns (more exactly 47, since one of the columns will be found to have nothing in it), numbered at top and bottom 0, 1, ..., 23, 0, 1, ..., 23. Each of the 48 columns is to be summed from bottom to top (not as for S in groups of 30), and the sums are to be entered in the form "Sums of series M." The 24 sums which come from the left half of the board will be entered in the row marked "red left," those from the right in the row marked "red right."

Lay the red N sheet on the other board, and transfer the strips.

In accordance with § 9, it will now usually suffice to sum only the columns appertaining to the even hours 0, 2, 4, ..., 22; as these hours are repeated twice, there will now be 24 columns to sum.

The sums are then to be entered on the form "Sums of series N," in the alternate columns. The complete form is provided, so that all the 24 hourly values may be used if it be thought desirable, but this labour seems unnecessary, at least in a long series of observations.

Lay the successive red sheets on the vacant board, transfer, sum, and enter, until all the red sheets are exhausted.

In the case of S, M, MS the sums of all the columns are necessary, but in the other eight arrangements only the sums of the alternate columns, those of the even hours, are usually necessary. For a short series of observations it may be best to use all the columns, but in this case it will certainly not be worth while to attempt the evaluation of ν, J, Q, μ, 2SM, which are all small in amount.

If the tides of long period MSf, Mf, Mm are required, the 24 numbers written on each strip must be added together, and the sum entered in the form "Long period tides—daily sums."

Clean the strips.

In exactly the same way work through the next 74 days, from 74^d to 74^d+73^d, with yellow guide sheets. Then clean the strips, and take another 74 days with green guide sheets, and so on with the blue and violet.

In the last (violet) set attention must be paid to the rules as to the places where the analysis is to stop in each arrangement.

If the year of observation is so incomplete that the hiatus cannot be made good by interpolation, or if the series does not run over the complete year, the series must stop with one of the days specified in the table at the end of § 7, and a note must be made of the number of days used in each series.

The strips marked for omission on the violet sheets, or those

selected for omission under the rules of § 7, may be hidden by a sheet of paper when the summations are being made.

The additions of S, M, MS may be verified by proving that the grand total of all the numbers (inclusive of omitted strips in S) written in each of the sets of 74 days is the same in whatever way they are arranged. Thus, the sum of the 48 columns should be equal to the sum of the daily sums. An incomplete verification in the other arrangements, when only half the columns are summed, is found by showing that the sum of all the hourly sums of each 74 days is nearly equal to half the grand total of all the numbers written in that period of 74 days.

When the guide sheets become worn with many pin pricks they may easily be patched with adhesive paper. There seems no reason why this patching should not go on almost indefinitely.*

* It is possible that it may be desired to evaluate the tides OO and 2N, for which no guide sheets are provided. I therefore, give instructions for the preparation of guide sheets for these cases. They will be understood by any one who has the set of guide sheets before him. With the instructions given below, the computer might indeed set up the strips without a guide-sheet.

I describe the staircase as descending from left to right or from right to left, and I define a short step as being one space down and one space to the left or right, as the case may be, and a long step as one space down and two to the left or right, as the case may be. When I say, for example, that a short follows 2, I mean that 2 to 3 is a short step. The first mark on each sheet is specified by its incidence in the row of hours at the top.

OO; descending from left to right.

The sequence is long several times repeated and then short.

Red; 0 between 0^h and 1^h; shorts follow 2, 7, 13, 19, 24, 30, 36, 41, 47, 53, 58, 64, 69.

Yellow; 0 between 15^h and 16^h; shorts follow 1, 7, 12, 18, 24, 29, 35, 41, 46, 52, 57, 63, 69.

Green; 0 between 6^h and 7^h; shorts follow 0, 6, 12, 17, 23, 29, 34, 40, 45, 51, 57, 62, 68.

Blue; 0 between 21^h and 22^h; shorts follow 0, 5, 11, 17, 22, 28, 33, 39, 45, 50, 56, 62, 67.

Violet; 0 between 11^h and 12^h; shorts follow 5, 10, 16, 21, 27, 33, 38, 44, 50, 55, 61, 67, 72.

The last strip used for a year is 72.

2N: descending from right to left.

The sequence is long, long, short, long, long, short, and at intervals three longs and a short.

Red; 0 between 22^h and 23^h; shorts follow 1, 4, ... , 16; 20, 23, ... , 35; 39, 42, ... , 57; 61, 64, ... , 70.

Yellow; 0 between 18^h and 19^h; shorts follow 2; 6, 9, ... , 21; 25, 28, ... , 40; 44, 47, ... , 59; 63, 66, ... , 72.

Green; 0 between 13^h and 14^h; shorts follow 1, 4; 8, 11, ... , 26; 30, 33, ... , 45; 49, 52, ... , 64; 68, 71.

Blue; 0 between 8^h and 9^h; shorts follow 0, 3, 6, 9; 13, 16, ... , 28; 32, 35, ... , 50; 54, 57, ... , 69; 73.

Hourly sums and harmonic analysis.

Complete the summations in the forms for hourly sums, and copy into the forms for harmonic analysis. In this copying it will generally suffice if the last figure in the hourly sums be omitted; for example, if the observations are entered to the nearest tenth of a foot the hourly sums will be given in the same unit, and it will suffice if the hourly sums analysed be written to the nearest foot.

There are 12 analyses (one for each month of 30 days) for the hourly sums in S, and one analysis for each of the other 10 arrangements. All the forms are provided with spaces for 24 hourly sums, but in the eight series N, L, ν, O, Q, J, μ, 2SM, where only 12 values will commonly be used, the entries will only be made on the alternate rows of 0^h, 2^h, ..., 22^h. In these cases the divisor 12, which occurs in the penultimate stage of finding the A's and B's, must be replaced by 6.

The large divisors (viz., 369 for M, μ, 2SM, MS; 369 for O; 358 for N, L, J, Q; and 350 for ν) represent the number of days under reduction, and must be altered appropriately (see table, § 7) if there be long gaps in the observations, or if the year be incomplete, or if the series be a short one.

If some one of the monthly analyses of S is deficient the missing $\mathfrak{A}$'s and $\mathfrak{B}$'s are to be made good by interpolation.

It is then necessary to analyse the monthly values of the $\mathfrak{A}$'s and $\mathfrak{B}$'s derived from the 12 analyses of S. We thus obtain A_0, A_1, B_1, A_2, B_2, C_0, c_0, C_1, D_1, c_1, d_1, E_0, e_0, E_1, F_2, e_1, f_1, E_2, F_2, e_2, f_2,. The rules for these analyses when the year is incomplete are given in § 4, and the computation forms only apply to the case of the complete year.

Astronomical data and final reduction.

Determine from the 'Nautical Almanac' and Major Baird's 'Manual of Tidal Observations'* the astronomical data at 0^h local M.T. on day 0, and proceed according to the form to find the initial arguments and factors for reduction. The astronomical data are then to be used in the forms for final reduction.

We have generally $B = R \sin \zeta$, $A = R \cos \zeta$; the forms are arranged so that colog A is to be added to log B to find log tan ζ, and thence ζ. If ζ lies between $-45°$ and $45°$ or between $135°$ and $225°$, log sec ζ is added to log A to find R; if ζ lies between $45°$ and $135°$ or between $225°$ and $315°$, log cosec ζ is added to log B to find R.

Violet; 0 between 4^h and 5^h; shorts follow 2, 5, ..., 14; 18, 21, ..., 33; 37, 40, ..., 52; 56, 59, ..., 71.

The last strip used for a year is 61.

* Taylor and Francis, London, 1886, price 7*s*. 6*d*.

Accordingly the computation form has log sec ζ, room being left for the syllable "co" if necessary; underneath this is written log , and the computer will insert A or B as the case may be.

There is usually required also a numerical factor 1/f or $\mathfrak{F}$, or both; the logarithms of the 1/f are found amongst the astronomical data, and the logarithms of the constant $\mathfrak{F}$'s are printed in the forms in their proper places.

The treatment of the 21 harmonic components derived from the harmonic analysis of the 5 $\mathfrak{A}$'s and $\mathfrak{B}$'s is shown in the forms.

The tides of long period.

The processes involved in the evaluation of these tides are sufficiently shown in the forms.

Postscript. December 17, 1892.

Correction to previous paper.

An error has been detected on p. 333 of my paper "On the Harmonic Analysis of Tidal Observations of High and Low Water," 'Roy. Soc. Proc.,' vol. 48 (1890).

In the example of reduction certain tabular values, extracted from Baird's 'Manual,' have the wrong signs attributed to them. Since in 1887 the longitude of the moon's node lay between 0° and 180°, the signs of ν, ξ, ν', $2\nu''$ should be positive, instead of negative as stated on p. 333. When this error is corrected the reduction leads to the following comparison:—

		Computed.	Mean of 9 yrs. obs.
		ft.	ft.
M_2	H =	3·98	4·04
	κ =	330°	330°
S_2	H =	1·68	1·63
	κ =	2°	3°
K_2	H =	0·46	0·41
	κ =	2°	352°
N	H =	1·04	1·00
	κ =	317°	313°
L	H =	0·11	0·09
	κ =	237°	308°
K_1	H =	1·24	1·40
	κ =	41°	45°
O	H =	0·69	0·66
	κ =	59°	48°
P	H =	0·41	0·40
	κ =	41°	43°

If the calculations had been conducted by rigorous methods, the two columns would have agreed nearly with one another.

I may mention that I have copies of a table of $(\gamma - \sigma) t$ up to 90 days (see p. 304 of the paper here referred to) which I shall be glad to give to any one actually engaged in the reduction of H. and L.W. observations.

II. "On some new Reptiles from the Elgin Sandstone." By E. T. NEWTON, F.G.S. Communicated by Sir ARCHIBALD GEIKIE, F.R.S. Received November 28, 1892.

(Abstract.)

During the last few years a number of Reptilian remains have been obtained from the Elgin Sandstone at Cuttie's Hillock, near Elgin, which are now in the possession of the Elgin Museum and of the Geological Survey. These specimens represent at least eight distinct skeletons, seven of which undoubtedly belong to the Dicynodontia, and one is a singular horned Reptile, new to science. All the remains yet found in this quarry are in the condition of hollow moulds, the bones themselves having entirely disappeared. In order, therefore, to render the specimens available for study, it was necessary, in the first place, so to display and preserve these cavities that casts might be taken which would reproduce the form of the original bones. Gutta-percha was found to be the most suitable material for taking these impressions; and in some instances, especially in the case of the skulls, the casts had to be made in several parts and afterwards joined together.

The first specimen described is named *Gordonia Traquairi;* it is the one noticed by Dr. Traquair in 1885, and referred to the Dicynodontia; besides the skull, it includes fragmentary portions of other parts of the skeleton, and is contained in a block of sandstone which has been split open so as to divide the skull almost vertically and longitudinally. The two halves have been so developed that casts made from them exhibit the left side and upper surface, as well as the main parts of the palate and lower jaw. In general appearance this skull resembles those of *Dicynodon* and *Oudenodon.* The nasal openings are double and directed laterally; the orbits are large and look somewhat forwards and upwards. The supra-temporal fossa is large, and bounded above by the prominent parieto-squamosal crest, and below by the wide supra-temporal bar, which extends downwards posteriorly to form the long pedicle for the articulation of the lower jaw. There is no lower temporal bar. The maxilla is directed downwards and forwards to end in a small tusk. Seen from above, the skull is narrow in the inter-orbital and nasal regions, but wide posteriorly across the temporal bars, although the brain-case itself is very

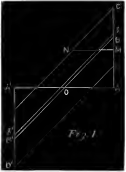


Fig. 1

Fig 2

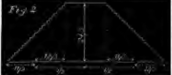


Fig 3.